By what we learned about the author and his book, we of course wish we could have had the opportunity to talk with him before we wrote the first two chapters in [3] and he wrote his nine chapters plus two appendices.

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Potential theory, an analytic and probabilistic approach to balayage, by J. Bliedtner and W. Hansen, Springer-Verlag, Berlin, Heidelberg, New York and Tokyo, 1986, xi + 434 pp., $40.00. ISBN 0-387-16396-4

Potential theory and probability theory began a symbiosis in the 1940s and 1950s which continues to yield some of the deepest insights into the two subjects. On the surface, they seem quite dissimilar; fundamentally, certain aspects are identical.

The genesis of modern potential theory was H. Cartan's investigation of Newtonian potential theory in the 1940s. If μ is a distribution in R^3, then the potential generated by μ is the function $U\mu(x) = \int |x-y|^{-1}\mu(dy)$. Some hint of the richness of this class of potentials rests in the observation that every positive superharmonic function in R^3 can be represented as the sum of a positive constant and the potential of a positive measure μ. This collection $\mathbf{S}$ of superharmonic functions is the potential cone of Newtonian potential theory: it is closed under addition and scalar multiplication, and the minimum of two functions in $\mathbf{S}$ is again in $\mathbf{S}$.

Many of the problems of potential theory are rooted in the problems of electrostatics in the classical case. Place a unit charge on a conductor B in R^3. The electrons will rush to the skin of B and assume an equilibrium distribution π so that the potential $U\pi(x)$ of this distribution is constant for x in the interior of B. We can obtain $U\pi(x)$ from $\mathbf{S}$ as follows. Let $f = \inf\{g \in \mathbf{S}: g \geqslant 1 \text{ on } B\}$. There is a unique element $U\gamma$ in $\mathbf{S}$ which agrees with f almost everywhere. The total mass of γ is called the Newtonian capacity of the conductor B and is denoted cap(B). Then $\pi = \gamma/\text{cap}(B)$ and $U\pi = U\gamma/\text{cap}(B)$. The function f is called the réduite of 1 on B, and $U\gamma$ is its regularization. More generally, the function $\inf\{g \in \mathbf{S}\colon\ g \geqslant U\rho \text{ on } B\}$ is called the réduite of $U\rho$ on B, and its regularization is called the balayage of $U\rho$ on B. This regularization has the form $U\nu$, and ν is called the balayage of ρ on B.

The balayage (or "sweeping") operation is the central one in potential theory. One well-known way in which it appears is in solving the Dirichlet problem. If D is a smooth open domain in R^3 and if f is a continuous function defined on its boundary, then a function h defined on D solves the Dirichlet problem for (D, f) provided h is harmonic on D and h agrees with f on the boundary. The following remarkable formula provides a solution. Let μ^x be the balayage of a unit point mass at x onto the boundary of D. Then

$$h(x) = \int f(y)\mu^x(dy).$$

The solution of this problem when D and f are not smooth is known as the Perron-Wiener-Brelot solution and is completely treated in Bliedtner and Hansen's Chapter VII.

Also in the 1940s, Kakutani found a probabilistic method for solving the Dirichlet problem. Let $X(t)$ be Brownian motion in R^3: this is the Markov process which simulates the movement of a pollen particle due to bombardment by water molecules. It is characterized as the stochastic process with continuous trajectories starting at a point x_0 so that

$$P^{x_0}[X(t_1) \in dx_1, \ldots, X(t_n) \in dx_n]$$
$$= p_{t_1}(x_0, x_1)p_{t_2-t_1}(x_1, x_2) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n)\,dx_1 \cdots dx_n,$$

where $0 \leqslant t_1 < \cdots < t_n$, and

$$p_t(x, y) = (2\pi t)^{-3/2}\exp\left(-|x-y|^2/2t\right).$$

Kakutani's method is the following. Let $T = \inf\{t > 0\colon X(t) \text{ is in the boundary of } D\}$: T is a random variable. Then $h(x) = \int f(X(T))\,dP^x$. That is, $\mu^x(dy) = P^x[X(T) \in dy]$: to obtain μ^x, sweep the point mass at x along the Brownian path to the first point where it hits the boundary. This correspondence between balayage and Brownian motion is no accident, and most of the formulas of potential theory can be interpreted probabilistically by relying on the fact that $\int p_t(x, y)\,dt$ is a constant times $|x-y|^{-1}$. One additional example which has been developed recently in the probability literature involves the remarkably useful last exit time $L = \sup\{t > 0\colon X(t) \text{ is in } B\}$. In this case, $P^x[X(L) \in dy; L > 0]$ is a constant times $|x-y|^{-1}\pi(dy)$, where π is the equilibrium measure of B.

Bliedtner and Hansen's volume is devoted to this interplay between potential theory and Markov processes, with balayage serving as the guiding theme. The correspondence between Newtonian potential theory and Brownian motion described above extends to more general situations: G. Hunt's articles in 1957–58 served as major illuminations of the subject. Bauer and Brelot developed harmonic spaces which are in correspondence with diffusions (Markov processes with continuous paths). Right continuous Markov processes correspond to standard balayage spaces, a subject developed in large part by the authors. Far from being mere generalizations of the Newtonian case, the development of these subjects has been proceeding apace out of necessity in both the potential theory and probability literature. That this subject is a good framework for the study of second-order elliptic and parabolic equations is made amply apparent in the excellent Chapter VIII on partial differential equations. The study of Newtonian potential theory is really the study of the Laplacian: $-\Delta U\mu$ is simply a multiple of μ. The inverse of a second-order elliptic or parabolic partial differential operator L can be used to generate a harmonic balayage space just as the Laplacian is used in R^3. Harmonic analysts also will find the material on convolution semigroups and Riesz potentials of interest.

This book is a volume in the Springer Universitext series and will serve well as a textbook in a graduate topics course. It begins with classical potential theory and other preliminaries and then two chapters on excessive and hyperharmonic functions. The authors introduce Markov processes in Chapter IV and provide a nice development for people who have been exposed to a modest amount of probability theory. They end Chapter IV by stating four equivalent views of potential theory: these are of both analytic and probabilistic type and set the tone of the book. The examples illustrating Markov processes in Chapter V constitute as nice a collection as I have seen in any text, although a dyed-in-the-wool probabilist may find the material more analytic than probabilistic. Chapters VI and VII focus on balayage theory and the Dirichlet problem and provide thorough and readable treatments of each subject. The authors have provided a page of Basic Notations, a three-page Index of Symbols, a comprehensive Subject Index and a cross-referenced Guide to Standard Examples. These aids will make the book doubly appreciated by graduate students learning the subject and by researchers who read about a particular topic. Bliedtner and Hansen have distilled the essence of balayage theory for the reader. Its richness lies in the interplay between analysis and probability, and it is just this which they have used to give an enjoyable presentation of this field of research.

With this book, balayage theory has reached the enviable situation of a topic which can be presented in this form and which still offers largely unexplored vistas. For example, semipolar sets (Chapter VI, §5) have stimulated many of the developments of the theory and continue to defy analysis. A semipolar (resp., polar) set is characterized by the fact that the associated Markov process will visit it at most countably often (resp., will never visit it). In Newtonian potential theory, every semipolar set is polar, and this is equivalent to the bounded energy principle: if μ is any signed measure so that $U|\mu|$ is bounded, then $\int U\mu(x)\mu(dx) \geqslant 0$. This equivalence is true quite generally, but in other potential theories there can be semipolar sets which are not polar. Even in the case of potential theories and Markov processes arising from convolution semigroups in R^d, little is known about these sets. It has been a conjecture for twenty years that, except in processes where a translation component interferes, every semipolar set is polar. This still remains to be decided.

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