However, as up-to-date as this volume is, the field is moving so quickly that a student will not find enough about the topics of interest in today's research. Representation theory dominates today, with spectacular achievements in the representation theory of the groups of Lie type and whole new areas starting up in general representation theory. The use now of representation theory as a tool for studying structure of simple groups is very minimal, though in the long run one suspects this will not remain the case. The other very active area of finite group theory is the study of more geometrical approaches. These ideas, in particular, the amalgam method introduced by Goldschmidt, have blossomed and have applications to structural questions. Indeed, some of the proofs of the basic "pushing up" theorems in local methods, which Suzuki exposits so well, are fast becoming obsolete due to these much more powerful geometric methods. The quickness of progress in finite group theory will no doubt continue to plague authors of books on the subject.

J. L. ALPERIN

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Introduction to various aspects of degree theory in Banach spaces, by E. H. Rothe, Mathematical Surveys and Monographs, vol. 23, Amer. Math. Soc., 1986, vi + 242 pp., $60.00. ISBN 0-8218-1522-9

Those who have not seen his name before or know as little about the author as we do will suspect that there must be something special about him, since his manuscript was published in an edition which is one of the finest we have seen in recent years. Golden letters on the cover and paper so innocently white that one hesitates to mark the only thing that has to be, namely the ends of proofs, which are often difficult to find since proofs are long, interrupted by lemmas with proofs, etc. The secret is easily brought to light if one starts reading as usual, i.e., references first. There one finds his first paper **[5]** on the subject, and a look into the original reveals that it was written in 1936. In other words the book appeared just in time to celebrate the golden wedding of author and topological degree.

Digging more into history we see that the fundamental paper **[4]** on degree theory in Banach spaces by J. Leray and J. Schauder was published in 1934, and from the second part of this paper it is obvious that the class of maps they consider was motivated by its usefulness in solving elliptic boundary value problems. In fact it was a revolutionary breakthrough in the treatment of these and other nonlinear problems, studied intensively and solely by the then almighty method of successive approximations. Since any revolution is based on previous evolution, let us note that *Leray-Schauder degree*, as it is called today, had a well-known forerunner, the corresponding concept for continuous maps on $\mathbf{R}^n$, called *Brouwer degree*, since L. E. J. Brouwer's paper of

1912. This is an outcome of combinatorial (or prealgebraic) topology which, together with some of its topological consequences and as a byproduct of intersection theory, was made popular by the then brand-new classic [**1**] of P. Alexandroff and H. Hopf of 1935. Clearly, the name *Banach space* indicates that some knowledge about infinite-dimensional normed vector spaces and analytical/topological concepts related to them is necessary too. Although most writers still did not feel so at home in such spaces as we do today, a flood of papers and ideas on (linear) functional analysis, as it is called in our days, was already canalized by S. Banach's classic [**2**] of 1932. Almost 20 years later papers of M. Nagumo and E. Heinz made evident that degree theory can also be established elegantly by means of purely analytical tools, and this is what the author tells us in his introduction: "the book is written from the point of view of an analyst." Pursuing this point of view, let us now sketch the subject and its role today.

Those who had a reasonable teacher in complex function theory know about the useful role played by the winding number $w(\Gamma, a)$ of plane piecewise differentiable curves Γ (with respect to points $a \notin \Gamma$) in the study of zeros of analytic functions. Those who even had a good teacher (or book) also know that the concept can be extended to continuous Γ, due to the observation that $w(\Gamma_1, a) = w(\Gamma_2, a)$ if $a \notin \Gamma$ and the C^1-curves Γ_1 and Γ_2 are sufficiently close to Γ, and that it is not difficult to establish its essential property, namely homotopy invariance. To get a concept of similar usefulness for the study of finite systems $f_i(x_1, \dots, x_n) = y_i$ (for $i = 1, \dots, n$), we simply imitate, consider open bounded $\Omega \subset \mathbf{R}^n$ instead of the regions enclosed by Γ, continuous maps $f\colon \overline{\Omega} \to \mathbf{R}^n$ and points $y \in \mathbf{R}^n \setminus f(\partial\Omega)$, and try to find a $\mathbf{Z}$-valued function d on these triples (f, Ω, y) which satisfies at least the following natural requirements:

(d_1) $d(\mathrm{id}, \Omega, y) = 1$ for $y \in \Omega$ ($\mathrm{id}(x) = x$ on $\mathbf{R}^n$).

(d_2) $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ if Ω_1 and Ω_2 are disjoint open subsets of Ω and $y \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$.

(d_3) $d(h(t, \cdot), \Omega, y(t))$ is independent of $t \in J = [0,1]$ if $h\colon J \times \overline{\Omega} \to \mathbf{R}^n$ and $y\colon J \to \mathbf{R}^n$ are continuous and $y(t) \notin h(t, \partial\Omega)$ on J.

Condition (d_2) implies that we have a solution of $f(x) = y$ if $d(f, \Omega, y) \neq 0$, and the homotopy invariance expressed by (d_3) is most useful in detecting $d(f, \Omega, y) \neq 0$. In particular, Brouwer's fixed point theorem, saying that a continuous f from the unit ball $\overline{B}_1(0)$ into itself has a fixed point, follows from (d_3) and (d_1).

In the construction of such a function d we find it most instructive to start at the end and to cook the problem down, step by step, to the simplest case $f(x) = Ax$ with $\det A \neq 0$, i.e., we go from $C(\overline{\Omega})$ to $C^\infty(\Omega) \cap C(\overline{\Omega})$ by means of (d_3) and the fact that $C^\infty(\Omega) \cap C(\overline{\Omega})$ is dense in $C(\overline{\Omega})$, then from singular values y to regular ones (i.e. $\det f'(x) \neq 0$ whenever $f(x) = y$) by means of (d_3) and an easy special case of Sard's Lemma, saying that $f(\{x \in \Omega\colon \det f'(x) = 0\})$ has n-dimensional Lebesgue measure zero, and finally, since we have at most finitely many solutions x^i in the regular case, we use (d_2) and (d_3) to see that computation of $d(g, B_1(0), 0)$ with $g(x) = f'(x^i)x$ is all we need. Now elementary linear algebra shows that this integer is necessarily given by sgn det $f'(x^i)$. The advantage of this first step is that on our way down we use only things which we need anyway on the way up, that we have proved uniqueness of d, and that our starting point, i.e. the definition

$$d(f,\Omega,y)=\sum_{x\in f^{-1}(y)}\operatorname{sgn}\det f'(x)$$

for the regular case, does not fall from heaven. On the way back to $C(\bar{\Omega})$ the only difficulty is getting rid of the regular case, i.e. to prove $d(f,\Omega,y^1)=d(f,\Omega,y^2)$ if y^1 and y^2 are regular and sufficiently close to $y\notin f(\partial\Omega)$ with $f\in C^2(\Omega)\cap C(\bar{\Omega})$, say. Instead of using an analytically awkward transversality argument by going one dimension higher as in this book, we prefer the simple trick of writing $d(f,\Omega,y^i)$ as

$$\int_\Omega \varphi_\varepsilon(f(x)-y^i)\det f'(x)\,dx$$

with mollifiers φ_ε of sufficiently small support. Then the desired equality follows easily from the divergence theorem and $\sum_{i=1}^n \partial\alpha_{ij}/\partial x_i=0$ (for $j=1,\dots,n$), where α_{ij} is the cofactor of $\partial f_j(x)/\partial x_i$ in $\det f'(x)$.

Now, let us replace $\mathbf{R}^n$ by an arbitrary real Banach space X. If $\dim X<\infty$ then the extension of Brouwer's degree just discussed is obvious, since X is linearly isomorphic to $\mathbf{R}^n$. In case $\dim X=\infty$ there cannot be a degree satisfying (d_1)-(d_3) for all continuous f since, for example, Brouwer's fixed point theorem is no longer valid. However, a unique degree exists for the subclass of maps $f=\mathrm{id}-f_0$ with f_0 continuous and $f_0(\bar{\Omega})$ relatively compact, considered by Leray and Schauder in [**4**] and by the author throughout the book. The idea behind its definition is simple, since f_0 can be approximated, uniformly on $\bar{\Omega}$, by finite-dimensional maps g (i.e., $g(\bar{\Omega})\subset Y$ with $\dim Y<\infty$) and, assuming $y\in Y$ without loss of generality, the solutions of $x-g(x)=y$ are already in $\Omega\cap Y$; in other words repeated use of (d_3) brings the problem down to the uniquely determined degree for finite-dimensional spaces and return to the general case presents no difficulties. Since this is so, we do not see any reason, at this level, to imitate the $\mathbf{R}^n$-procedure sketched above, i.e., to start with the regular case under the additional assumption that f_0 be C^2, say, and to check how far one can go this way, as is done in this book based on the author's paper [**6**]. Also a newcomer may suspect that at the end we cannot get rid of a C^1-assumption unless we consider very special X, and he may complain about the starting point, i.e. the analogue of $d(g,B_1(0),0)=\operatorname{sgn}\det A$ for $g(x)=Ax$, which is not motivated and requires a lot of preparation, while it is so easy to deduce from the elementary spectral theorem for compact linear operators once the degree is established in the quick way.

Of course the development did not stop at this point. One has considered more general settings, say manifolds modelled on X or absolute neighborhood retracts, and more general classes of maps, say set-contractions or multivalued maps. On the other hand many different applications called for invention of new or improvement of other old methods, so that degree (or index) theory still plays a useful role, but no longer the dominant one it had for some period in the abstract treatment of nonlinear problems, which we may call nonlinear functional analysis today.

By what we learned about the author and his book, we of course wish we could have had the opportunity to talk with him before we wrote the first two chapters in [3] and he wrote his nine chapters plus two appendices.

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Potential theory, an analytic and probabilistic approach to balayage, by J. Bliedtner and W. Hansen, Springer-Verlag, Berlin, Heidelberg, New York and Tokyo, 1986, xi + 434 pp., $40.00. ISBN 0-387-16396-4

Potential theory and probability theory began a symbiosis in the 1940s and 1950s which continues to yield some of the deepest insights into the two subjects. On the surface, they seem quite dissimilar; fundamentally, certain aspects are identical.

The genesis of modern potential theory was H. Cartan's investigation of Newtonian potential theory in the 1940s. If μ is a distribution in R^3, then the potential generated by μ is the function $U\mu(x) = \int |x - y|^{-1}\mu(dy)$. Some hint of the richness of this class of potentials rests in the observation that every positive superharmonic function in R^3 can be represented as the sum of a positive constant and the potential of a positive measure μ. This collection **S** of superharmonic functions is the potential cone of Newtonian potential theory: it is closed under addition and scalar multiplication, and the minimum of two functions in **S** is again in **S**.

Many of the problems of potential theory are rooted in the problems of electrostatics in the classical case. Place a unit charge on a conductor B in R^3. The electrons will rush to the skin of B and assume an equilibrium distribution π so that the potential $U\pi(x)$ of this distribution is constant for x in the interior of B. We can obtain $U\pi(x)$ from **S** as follows. Let $f = \inf\{g \in \mathbf{S}: g \geqslant 1 \text{ on } B\}$. There is a unique element $U\gamma$ in **S** which agrees with f almost everywhere. The total mass of γ is called the Newtonian capacity of the