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GENERALIZED ALBANESE VARIETIES FOR SURFACES

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In this paper we announce a solution to the generalized Albanese problem for smooth projective surfaces. More precisely, for such a surface X over a field k and for each modulus m (see next paragraph) we show the existence of a pair (G, α), where G is a commutative algebraic group over k (or more generally a principal homogeneous space under such a group), $\alpha\colon X \to G$ is a rational map, and any rational map with modulus m factors through α.

Let X be such a surface and let $U = X \sim \bigcup D_j$ be the complement of a finite number of integral divisors on X. In [**2**, Chapter 3, Proposition 1] it was shown that for a rational map $\alpha\colon X \to G$ into an algebraic group we get a homomorphism $\gamma_m\colon C_m(X) \to G(k)$ for some modulus m, where $C_m(X)$ denotes the K-theoretic idele class group of X. When domain$(\alpha) = U$ we have $m = \sum m_j D_j$ with $m_j \geq 1$. In this situation we say that α admits m as modulus.

It is clear that by usual descent arguments we may assume that k is algebraically closed and work with algebraic groups rather than principal homogeneous spaces.

Let Cat_m denote the category of maps $\alpha\colon X \to G$ which admit m as modulus.

THEOREM 1. *In* Cat_m *there exists* $\alpha\colon X \to G_{um}$ *with the universal mapping property described above.*

SKETCH OF THE PROOF. By [**5**, Corollary to Theorem 2] it suffices to show that the dimension of algebraic groups G with $\beta\colon X \to G$ in Cat_m and β maximal [**5**, Definition 2] is bounded. For this by blowing up points in U we reduce to the case of a Lefschetz pencil $\pi\colon X' \to \mathbf{P}^1$ with m flat over $\mathbf{P}^1$.

Then by using [**2**, Chapter 3, Lemma 1] we see that $(\beta, \pi)\colon X'' \to G \times S$ admits m as a modulus in the sense of [**6**, Definition 1] ($X'' \to S \subset \mathbf{P}^1$ is the smooth part of the pencil). Hence it factors through the relative generalized jacobian J_m of X'' [**6**, Theorem 1]. Then it is easy to see that the dimension of the group generated by β is equal to the dimension of the image of the composite map

$$J_m \to G \times S \overset{\mathrm{proj}}{\to} G.$$

Therefore if β generates G then $\dim(G) \leq \dim(J_m)$.

REMARK. We can give an alternate proof of Theorem 1 by applying [**7**, §3, Proposition 4] to show that α admits m as modulus iff

$$\alpha^*(\Omega_G^{\mathrm{inv}}) \subset (H^0(U, \Omega_U)^{d=0} \cap H^0(X, \Omega_X(-m))).$$

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This ties Theorem 1 with the modulus defined by Faltings and Wüstholz [**1**, Theorem 1] in characteristic zero.

For the construction of the pair (G_{um}, α) we have

THEOREM 2. *In characteristic zero, the universal pair can be constructed by rigidifying the Picard functor* $\mathrm{Pic}_{\mathrm{Pic}^0_X}$ *of the Picard variety* Pic^0_X *of* X.

SKETCH OF THE CONSTRUCTION. We know that G_{um} must be an extension of the Albanese variety Alb_X of X by a connected algebraic group. Therefore it comes from a rigidification of $\mathrm{Pic}_{\mathrm{Pic}^0_X}$ [**3**] (for rigidification see [**4**, Definition 2.1.1]). The rigidifier R is supported on $\{x_1, \ldots, x_r, 0\}$ in Pic^0_X, where $x_1, \ldots, x_r$ is a set of free generators for the image of

$$\mathrm{Kernel}(ZD_1 + \cdots + ZD_n \to \mathrm{Pic}_X(k) \to \mathrm{Pic}_X(k)/\mathrm{Pic}^0_X(k))$$

and 0 is the zero section. For a given m we can determine R explicitly (for a special case see [**3**]). Then α is obtained simply by using the definition of the rigidified Picard functor.

REMARKS. (1) For $m' \geq m$ we have an affine morphism $G_{um'} \to G_{um}$, hence $\varprojlim G_{um}$ exists. This pro-smooth group is important for the class-field theory of X.

(2) The homomorphism γ_m: $C_m(X) \to G_{um}(k)$ is surjective because for x in U, 1 in $C_m(x)$ is mapped to $\alpha(x)$ and α generates G_{um}. In characteristic zero it seems natural to expect that when we restrict to the idele classes of degree zero, γ_m is an isomorphism iff $p_g(X) = 0$.

The details together with the discussion of the relative case and the extension to dimensions > 2 will appear elsewhere.

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