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THE INDECOMPOSABLE K_3 OF FIELDS

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In this note, we describe an extension of Hilbert's Theorem 90 for K_2 of fields to the relative K_2 of semilocal PID's containing a field. Most of the results for K_2 of fields proven in [**M-S** and **S**] then carry over to the relative K_2 of semilocal PID's containing a field, e.g. computation of the torsion subgroup, and the norm residue isomorphism. Applying this to the semilocal ring of $\{0,1\}$ in $\mathbf{A}^1_E$, for a field E, gives a computation of the torsion and co-torsion in

$$K_3(E)^{\mathrm{ind}} := K_3(E)/K_3^M(E).$$

Specifically, we have

(1) The torsion subgroup of $K_3(E)^{\mathrm{ind}}$ is $H^0_{\text{ét}}(E, \mu_\infty^{\otimes 2})$.

(2) $K_3(F, \mathbf{Z}/n)^{\mathrm{ind}} \xrightarrow{\sim} H^1_{\text{ét}}(E, \mu_n^{\otimes 2})$ for $(n, \mathrm{char}(E)) = 1$, so

$$\varprojlim_n K_3(E)^{\mathrm{ind}}/l^n \xrightarrow{\sim} H^1_{\text{ét}}(E, \mathbf{Z}_l(2)) \quad \text{for } l \text{ prime, } l \neq \mathrm{char}(E).$$

(3) $K_3(E)^{\mathrm{ind}}$ satisfies Galois descent for extensions of degree prime to $\mathrm{char}(E)$.

(4) Bloch's group $B(E)$ is uniquely n-divisible if E contains an algebraically closed field, and $(n, \mathrm{char}(E)) = 1$.

The results (3) and (4) follow directly from (1) and (2). To prove (1) and (2), the essential case is when E is a finite extension of the prime field; when E has positive characteristic (1) and (2) follow from Quillen's computation of the K-theory of finite fields [**Q2**]. For E a number field, (1) and (2) are the conjectures of Lichtenbaum and Quillen in the case of K_3, i.e. if E is a number field, the Chern class

$$c_{2,1}\colon K_3(E)^{\mathrm{ind}} \otimes \mathbf{Z}_l \to H^1_{\text{ét}}(E, \mathbf{Z}_l(2))$$

is an isomorphism. Merkurjev and Suslin have obtained these results, using similar methods. Here we give a sketch of the proof of Hilbert's Theorem 90 for relative K_2, and its application to the Lichtenbaum-Quillen conjecture for K_3.

Let R be a semilocal PID with Jacobson radical I. Let $\mathcal{D}$ be an Azumaya algebra over R, and X the associated Brauer-Severi scheme over R. Let $\overline{X}$ denote the fiber over $\overline{R} := R/I$. There is an E_1 spectral sequence converging to the relative K-theory $K_*(X, \overline{X})$ analogous to the Quillen spectral sequence converging to $K_*(X)$; the E_2 term $E_2^{p,q}(X, \overline{X})$ is a relative analogue

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to $H^p(X, \mathcal{K}_{-q})$. One proves essentially as when R is a field:

LEMMA 1. *Assume that $\overline{\mathcal{D}}$ is split, and $\mathcal{D}$ has prime rank l over R. Let $h\colon R \to R'$ be a finite étale extension. Then*

$$E_2^{1,-2}(X,\overline{X}) \to E_2^{1,-2}(X_{R'},\overline{X}_{R'})$$

is injective.

Let S be a semilocal PID with Jacobson radical J. Let L be the quotient field of S. Weibel [**W**] has shown that $K_2(S,J)$ is generated by symbols $\{a,b\}$, with $a \in (1+J)^*$, $b \in L^*$. Suppose that S contains a field k containing $\boldsymbol{\mu}_l$, l a prime. Let α be in S^*, let $S^\alpha = S[X]/X^l - \alpha$, if $\mathrm{char}(k) \neq l$; if $\mathrm{char}(k) = l$, let $S^\alpha = S[X]/X^l - X - \alpha$. Let J^α be the Jacobson radical JS^α of S^α, let

$$N : K_*(S^\alpha, J^\alpha) \to K_*(S,J)$$

be the norm map, and let σ be a generator of $\mathrm{Gal}(S^\alpha/S)$.

LEMMA 2. $\{y, 1-N(y)\}$ *is in* $(1-\sigma)K_2(S^\alpha, J^\alpha)$ *for all* $y \in (1+J^\alpha)^*$.

PROOF (SKETCH). In [**S**], this is done by an easy direct computation. We proceed here by a generic element method.

Let F_0 be the prime field. If $F_0 = \mathbf{Q}$, let $R = \mathbf{Q}(\varsigma_l)[t]_{(t)}$; if $F_0 = \mathbf{F}_p$, let $R = \mathbf{F}_p(\varsigma_l, t_0)[t]_{(t)}$, with t_0 and t indeterminants. If E is an extension ring of F_0 let $R_E = E[t]_{(t)}$. We let k_0 be the ground field $\mathbf{Q}(\varsigma_l)$ or $\mathbf{F}_p(\varsigma_l, t_0)$. If T is an R-scheme let $\overline{T}$ denote the fiber over $\overline{R} := R/(t)$. After making a purely transcendental extension, we may assume that S contains k_0.

Let $x_0, \ldots, x_{l-1}$, v be indeterminants over k, and let $u = v^l$ if $l \neq p$; if $l = p$, let $u = v^p - v$. Let A^0, A, and B be the rings

$$A^0 = k_0[x_0, \ldots, x_{l-1}], \qquad A = A^0[u], \qquad B = A^0[v],$$

so $B = A[v]$. Let x be the element

$$x = 1 + t\sum x_i v^i \in R_B,$$

so x is the "generic element" of the universal Kummer extension (or Artin-Schreier extension if $l = p$) R_B/R_A with $x \equiv 1 \bmod t$.

Let $N\colon R_B \to R_A$ be the norm, σ the generator of $\mathrm{Gal}(R_B/R_A)$ with $\sigma(v) = \varsigma v$ for $l \neq p$, $\sigma(v) = v+1$ for $l = p$. Let $X^{1/l} = \mathrm{Spec}(R_B)$, $X = \mathrm{Spec}(R_A)$ and $X^0 = \mathrm{Spec}(R_{A^0})$. Let W be the closed subscheme of $X^{1/l}$ defined by the ideal $((1-N(x))/t)$, W' the subscheme defined by (x).

The symbol $\{x, 1-N(x)\}$ defines an element of $K_2(X^{1/l} - (W \cup W'), \overline{X}^{1/l} - \overline{W})$.

There is an affine open subset U of $X^{1/l} - W - W'$, containing the generic point of $\overline{X}^{1/l}$, and an element μ of $K_2(U, \overline{U})$ with

$$(*) \qquad \{x, 1-N(x)\} = \mu^\sigma/\mu \quad \text{in } K_2(U,\overline{U}).$$

This is fairly easy to show, the essential points being

(1) the inclusion $\overline{X}^{1/l} \to X^{1/l}$ is split,
(2) $X^{1/l}$ and X are both affine lines over X^0.

Given an element y of S^α, if y is sufficiently general, we can pull back the relation (*) to show that $\{y, 1-N(y)\} = z^\sigma/z$ in $K_2(S^\alpha, J^\alpha)$. We then conclude by a specialization argument. □

Let $\{S_i \mid i \in I\}$ be a filtering direct system of semilocal PID's. Let J_i be the Jacobson radical of S_i, and let S_∞ and J_∞ denote the direct limits

$$S_\infty = \varinjlim S_i, \qquad J_\infty = \varinjlim J_i.$$

We suppose that $\{S_i \mid i \in I\}$ satisfies

(I) Every x in $1 + J_\infty$ is a norm from S_∞^α.

(II) If $P(u)$ is a separable polynomial with coefficients in S_∞ and has degree $d < l$, then $P(u)$ factors completely in $S_\infty[u]$.

LEMMA 3. *Assuming* (I) *and* (II), *the quotient group*

$$K_2(S_\infty^\alpha, J_\infty^\alpha)/(1-\sigma)K_2(S_\infty^\alpha, J_\infty^\alpha)$$

is generated via symbols by $(1 + J_\infty^\alpha)^* \otimes L_\infty^*$.

The proof is essentially the same as the proof of the similar fact for K_2 of fields in [**B**–**T**].

THEOREM 1 (HILBERT'S THEOREM 90 FOR RELATIVE K_2). *Let S be a semilocal PID containing a field k, and containing an lth root of unity, l a prime. Let J be the Jacobson radical of S and α a unit in S. Let σ be a generator of* $\mathrm{Gal}(S^\alpha/S)$. *Then the complex*

$$K_2(S^\alpha, J^\alpha) \underset{(1-\sigma)}{\rightarrow} K_2(S^\alpha, J^\alpha) \underset{\text{Norm}}{\rightarrow} K_2(S, J)$$

is exact.

Using the above lemmas, the proof follows the same outline as Suslin's proof of Hilbert's Theorem 90 for K_2 of fields in [**S**].

Exactly as in Suslin [**S**], applying Hilbert's Theorem 90 to the generic Kummer extension $S(u^{1/l})/S(u)$, and the generic Artin-Schreier extension $S(\wp^{-1}(u))/S(u)$ one gets

THEOREM 2. *Let (S, J) be as above. The l-torsion subgroup of $K_2(S, J)$ is generated by symbols $\{f, \varsigma\}$, where f is in $(1+J)^*$, and ς is an lth root of unity. $K_2(S, J)$ is p-torsion free if k has characteristic $p > 0$.*

COROLLARY. *Let E be a field. Then the l-torsion subgroup of $K_3(E)^{\mathrm{ind}} := K_3(E)/K_3^M(E)$ is cyclic.*

PROOF. We may assume that E contains $\boldsymbol{\mu}_l$. Let (R, J) be the semilocal ring of $\{0, 1\}$ on $\mathbf{A}_E^1$. We have the exact sequence

$$0 \rightarrow K_3(E)^{\mathrm{ind}} \rightarrow K_2(R, J) \rightarrow K_2(R) \rightarrow .$$

From this and Theorem 2, it follows that ${}_lK_3(E)^{\mathrm{ind}}$ is generated by symbols of the form $\{f, \varsigma\}$, $f \in (1+J)^*$ with $f \in (R^*)^l$. Writing such an f as $f = g^l$, $g \in R^*$, we normalize g so that $g(0) = 1$. Then the class of f $\mathrm{mod}((1+J)^*)^l$ is determined by the value $g(1) \in \boldsymbol{\mu}_l$, proving the corollary. □

Now we can show

THEOREM 3. *Let E be a number field. The Chern class*

$$c_{2,1}\colon K_3(E)^{\mathrm{ind}}\otimes \mathbf{Z}_l \to H^1(E,\mathbf{Z}_l(2))$$

is an isomorphism, so the l-primary torsion in $K_3(E)^{\mathrm{ind}}$ is isomorphic to $H^0(E,(\boldsymbol{\mu}_l^{\infty})^{\otimes 2})$.

PROOF. We may assume that E contains $\boldsymbol{\mu}_l$. From [**Q**], $K_3(E)$ is finitely generated. From the above, the l-torsion in $K_3(E)^{\mathrm{ind}}$ is cyclic, hence the l-primary torsion is also cyclic. By [**B-T**], $K_3^M(E)$ is a 2-torsion group; by [**B**] the rank of $K_3(E)$ is r_2. Thus $K_3(E)^{\mathrm{ind}}/l$ is a $\mathbf{Z}/l$ vector space of dimension between r_2 and $1+r_2$. In addition, the Chern class vanishes on the Milnor K_3 (this follows from the integral product formula for Chern classes).

Let symb $H^1(E,\boldsymbol{\mu}_l^{\otimes 2})\to {}_lK_2(E)$ be the map

$$H^1(E,\boldsymbol{\mu}_l^{\otimes 2})\tilde{\to}(E^*/(E^*)^l)\otimes\boldsymbol{\mu}_l\to {}_lK_2(E)$$

and let H be the kernel of symb. Tate [**T**] has shown that H is $(\mathbf{Z}/l)^{1+r_2}$ and that symb is surjective. Soule [**So**] has shown that $c_{2,1}$ is surjective. Suslin shows in [**S**] that $H=c_{2,1}(K_3(E))$, and that the induced map

$$\bar{c}_{2,1}\colon {}_lK_2(E)\to H^1(E,\boldsymbol{\mu}_l^{\otimes Z})/H$$

is inverse to symb. This, together with the computation of $K_3(E)^{\mathrm{ind}}/l$ above, implies that the Chern class map

$$(*)\qquad c_{2,1}\colon K_3(E,\mathbf{Z}/l)^{\mathrm{ind}}\to H^1(E,\boldsymbol{\mu}_l^{\otimes 2})$$

is an isomorphism. Let R be as in the corollary, $\pi^*\colon E\to R$ the inclusion. A localization argument together with $(*)$ shows that

$$c_{2,1}\colon K_3(R;\mathbf{Z}/l)^{\mathrm{ind}}\to H^1(R,\boldsymbol{\mu}_l^{\otimes 2})$$

is surjective. We have the commutative square

$$(**)\qquad \begin{array}{ccc} K_3(R;\mathbf{Z}/l) & \overset{\delta_K}{\to} & K_3(E;\mathbf{Z}/l)\\ c_{2,1}\downarrow & & \downarrow c_{2,1}\\ H^1(R,\boldsymbol{\mu}_l^{\otimes 2}) & \overset{\delta_H}{\to} & H^1(E,\boldsymbol{\mu}_l^{\otimes 2}) \end{array}$$

where the δ's are the maps "reduce mod J" followed by the difference map. This diagram, together with the surjectivity of $c_{2,1}$ and δ_H, then implies that δ_K is surjective (δ_K is obviously surjective on the Milnor K_3), and hence $K_2(R,J;\mathbf{Z}/l)\to K_2(R;\mathbf{Z}/l)$ is injective. Thus $K_2(R,J)/l\to K_2(R)/l$ is injective.

Let L be the quotient field of R and $i\colon \mathrm{Spec}(L)\to\mathrm{Spec}(R)$ the inclusion. We have the commutative ladder

$$(***)\qquad \begin{array}{ccccccccc} K_3(E)^{\mathrm{ind}}/l^n & \to & K_2(R,J)/l^n & \to & K_2(R)/l^n & \to & (K_2(E)/l^n)^2 & \to & 0\\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H^1(E,\boldsymbol{\mu}_{l^n}^{\otimes 2}) & \to & H^2(R,i_!(\boldsymbol{\mu}_{l^n}^{\otimes 2})) & \to & H^2(R,\boldsymbol{\mu}_{l^n}^{\otimes 2})) & \to & (H^2(E\boldsymbol{\mu}_{l^n}^{\otimes 2}))^2 & \to & 0. \end{array}$$

the horizontal lines coming from the relativization sequence, and the vertical arrows Chern classes (Galois symbols). For all n, the Galois symbols for

$K_2(R)/l^n$ and $K_2(E)/l^n$ are isomorphisms. The surjectivity of δ_H shows that $H^2(R, i_!(\mu_l^{\otimes 2})) \to H^1(R, \mu_l^{\otimes 2})$ is injective, hence the second vertical arrow is an isomorphism for $n = 1$. We have the commutative ladder

$$\begin{array}{ccccccccc} {}_lK_2(R,J) & \to & K_2(R,J)/l^n & \to & K_2(R,J)/l^{n+1} & \to & K_2(R,J)/l & \to & 0 \\ \text{symb}\uparrow & & \downarrow & & \downarrow & & \downarrow & & \\ H^1(R, i_!(\mu_l^{\otimes 2})) & \to & H^2(R, i_!(\mu_{l^n}^{\otimes 2})) & \to & H^2(R, i_!(\mu_{l^{n+1}}^{\otimes 2})) & \to & H^2(R, i_!(\mu_l^{\otimes 2})) & & \end{array}$$

with the second row exact, and the first row exact except possibly at $K_2(R, J)/l^n$. This and induction show that the Galois symbol for $K_2(R, J)/l^n$ is an isomorphism for all n.

From the localization sequence on $\mathbf{A}_E^1$, together with a knowledge of $K_2(E)$, and K_1 of number fields, it follows that $K_2(R)\{l\}$ has no l-divisible subgroups, hence the same for $K_2(R, J)$. Thus for n sufficiently large, the l-primary torsion in $K_3(E)^{\text{ind}}$ injects into $K_2(R, J)/l^n$. From the ladder (***), it follows that the Chern class $c_{2,1} : K_3(E)^{\text{ind}} \to H^1(E, \mu_{l^n}^{\otimes 2})$ is injective on the l-primary torsion for large n. From this, the surjectivity of $c_{2,1}$, and the computation of the ranks of $K_3(E)^{\text{ind}}$ and $H^1(E, \mathbf{Z}_l(2))$ (the latter due to Tate [**T**]) it follows that the Chern class gives an isomorphism on the limits

$$c_{2,1}\colon K_3(E)^{\text{ind}} \otimes \mathbf{Z}_l \to H^1(E, \mathbf{Z}_l(2)),$$

proving the theorem. □

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