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CONSTANT MEAN CURVATURE SURFACES IN EUCLIDEAN THREE-SPACE

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1. Introduction. A soap film in equilibrium between two regions of different gas pressure is characterized mathematically by the condition that the surface it defines has nonzero constant mean curvature. It is an old problem in classical differential geometry to decide which finite topological types of surfaces can be realized as complete, properly immersed surfaces of nonzero constant mean curvature in E^3.

For a long time, the only known examples of such surfaces were, besides the round sphere and the cylinder, a family of rotationally invariant surfaces discovered in 1841 by Delaunay [**D**]. Alexandrov [**A**] proved that the only closed embedded surfaces of constant mean curvature are round spheres. More recently, H. C. Wente [**W**] constructed immersed tori of constant mean curvature. (Higher-dimensional examples had been constructed earlier by Wu-Yi Hsiang [**H**].)

The finite topological types of surfaces as above can be classified by the genus g and the number of ends m. We prove the following [**K1**, **K2**].

THEOREM 1. *Given any $g \geq 0$, $m \geq 3$, or $g \geq 2$, $m = 2$, there are infinitely many cmc surfaces of genus g and with m ends. (We adopt the abbreviation cmc surfaces to stand for properly immersed complete surfaces in E^3 of constant mean curvature* 1.) *The ends of these surfaces are asymptotic with exponential rate to Delaunay surfaces.*

THEOREM 2. *If $g \geq 3$ then there are infinitely many closed cmc surfaces of genus g.*

These theorems are a consequence of two theorems which provide general constructions for such surfaces. We refer the reader to [**K1**, **K2**] for the technical details.

Also doubly or triply periodic embedded cmc surfaces are constructed. These surfaces have the symmetries of some lattice in E^3 and our construction yields many additional surfaces beyond those which are known [**L**].

Many of the cmc surfaces of finite topological type are proved to be embedded, and work is still in progress to determine exactly for which (g, m) embedded examples exist.

Sketch of the proof. The Delaunay family of surfaces can be parametrized by a continuous parameter τ such that we have embedded surfaces for $\tau < 0$ and immersed ones for $\tau > 0$. The positive curvature regions tend to

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unit spheres as $\tau \to 0$ and the negative curvature regions enlarged by a factor of $|\tau|^{-1}$ tend to standard catenoids. Complete surfaces can be constructed by attaching pieces of Delaunay surfaces to (pieces of) unit spheres. On such a surface $\tilde{M}$ the mean curvature satisfies $H = 1 + \delta H$, where $\delta H = O(\tilde{\tau})$ and is supported on the regions of transition between the Delaunay pieces and the unit spheres. $\tilde{\tau}$ is the largest absolute value of the τ parameter of a Delaunay piece contained in $\tilde{M}$. We want to find a smooth function z on $\tilde{M}$ such that the image of $f + z\xi: \tilde{M} \to E^3$ is a cmc surface, where $f: \tilde{M} \to E^3$ is the immersion of $\tilde{M}$ in E^3 and ξ its Gauss map.

The linearized equation for z is $(\Delta_h + 2)z = -4|A|^2\delta H$ where A is the second fundamental form of $\tilde{M}$ and $h = \frac{1}{2}|A|^2 g$, g being the induced metric on $\tilde{M}$. As $\tilde{\tau} \to 0$ both the positive and negative curvature regions of $\tilde{M}$ with respect to the h metric tend to round spheres. This allows us to understand the spectrum of $L_h \equiv \Delta_h + 2$ on $\tilde{M}$. In particular we see that L_h has small eigenvalues corresponding to what we call approximate kernel. We need $-4|A|^2\delta H$ to be "orthogonal" to the approximate kernel. This amounts to the following "balancing condition": At each of the spheres used in constructing $\tilde{M}$ consider the unit vectors pointing in the direction of the Delaunay pieces attached to this sphere. We request that the sum of these vectors at each sphere is 0, where the sum is weighted with weights depending only on (and roughly proportional to) the τ parameter of the corresponding Delaunay piece.

We try to "invert" L_h now as follows: We define three suitable functions at each positive or negative curvature region of $\tilde{M}$, which functions are far away from being orthogonal in L^2 to the approximate kernel. We call their span the "substitute kernel" and we prove the following: Given a (smooth enough, bounded) function v on $\tilde{M}$ there is a function u such that $L_h u - v = w$ is an element of the substitute kernel. Assume for a moment that there is no approximate kernel, as happens if we have a doubly periodic surface with a lot of symmetry, for example. Then we would like to prove that if $\tilde{\tau}$ is small enough, there is a (small) ϕ such that $z+\phi$ solves the nonlinear problem. To do so we obtain suitable estimates for u and for the error of linearization and then we use the Schauder fixed point theorem to solve the problem. Note that these estimates are borderline in the sense that while $\delta H = O(\tilde{\tau})$ and $|A| = O(\tilde{\tau}^{-1})$, we need $z|A| = o(\tilde{\tau})$ for our scheme to work. An extra difficulty is that we have to define a new metric to obtain suitable higher-order (Schauder-type) estimates.

In general, we do have an approximate kernel and we deal with it as follows: We construct a family of surfaces F around $\tilde{M}$. This family is parametrized by small elements w of the substitute kernel decaying sufficiently fast at infinity. On the surface $M \in F$ corresponding to w we have $L_h u + 4|A|^2\delta H = w$. (We identify the function spaces on $\tilde{M}$ and M.) Then we apply the Schauder fixed point theorem on a suitable set of (w, ϕ)'s to prove what we want.

The above construction settles the cases $m \geq [g/2] + 3$. For small m's we have to work harder. We illustrate our approach by considering a specific construction in the case $g = 3$, $m = 0$. Consider the graph consisting of an equilateral triangle and three line segments connecting the center of the triangle with its vertices. We would like to construct an $\tilde{M}$ as above made of four spheres, each centered at one vertex, and six Delaunay pieces connecting the spheres and corresponding to the edges of the graph. For simplicity we (would) impose the maximum possible symmetry on the construction and then try to perturb as above to obtain what we want. This approach fails to work because the lengths of the Delaunay pieces would not match up in general to make a closed surface. (The balancing condition can be handled.) So we consider the universal cover of the above graph, still with maximum symmetry but with length of the sides different from the length of the other edges. We build an $\tilde{M}$ around this graph and around it a family F as above. Our family now has an extra degree of freedom: the τ-parameter of the surfaces is allowed to vary in some suitable interval around $\tilde{\tau}$. The surfaces $M \in F$ fail to close up by a certain period $l(M) \in \mathbf{R}$. By studying $l(M)$ we prove that if we choose $\tilde{\tau}$ small enough and then the lengths of the Delaunay pieces suitably large, we can modify the application of the Schauder fixed point theorem above to obtain an $M \in F$ which can be perturbed along the normal to give a cmc surface and which has $l(M) = 0$, hence producing the desired surface. The estimates on $l(M)$ are based on the fact that if $2p(\tau)$ is the period of the Delaunay surface of parameter τ, then

$$\frac{1}{\log|\tau|}\frac{dp}{d\tau} \to 1 \quad \text{as } \tau \to 0.$$

(Refer to [**K1**] for the precise definition of τ.)

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NOTE ADDED IN PROOF. The following necessary conditions for the existence of finite type embedded cmc surfaces have been established: Each end is contained in a cylinder and $m \neq 1$, by W. Meeks. The ends are asymptotically Delaunay and they satisfy a balancing condition, and $m = 2$ implies Delaunay, by N. Korevaar, R. Kusner, and B. Solomon.

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