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ON THE SITUATION OF NODES OF PLANE CURVES

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1. Introduction. We consider complex plane algebraic curves with nodes (i.e., ordinary double points). Such a curve is said to be nodal if it has only nodes as singularities. Salmon proposed the following problem: Describe the situation of nodes of an irreducible nodal curve ([**4**, Art. 45], [**2**, pp. 389–393]). Let n denote the degree of a nodal curve and d the number of nodes. The problem is trivial if $n \leq 6$ and $d \leq 8$. The first nontrivial case, $(n,d) = (6,9)$, has been analyzed by Halphen (cf. [**2**, p. 390]). The case

$$d \leq \min\{n(n+3)/6, (n-1)(n-2)/2\} \quad \text{and} \quad (n,d) \neq (6,9)$$

was investigated by Arbarello and Cornalba [**1**, Theorem 3.2]; we give another proof (see Proposition 3(i)). We consider the remaining cases, which are particularly important as they have applications to the moduli variety of curves.

Let $V_{n,d}$ be the variety of irreducible nodal curves of degree n with d nodes. *For* $n(n+3)/6 \leq d \leq (n-1)(n-2)/2$ *and* $(n,d) \neq (6,9)$, *we prove that the map* $p_d: V_{n,d} \to \mathrm{Sym}^d(\mathbf{P}^2)$, *which sends a curve to the set of its nodes, is a birational morphism onto its image* (Theorem, Part (i)) *and give a rough description of the image* (Corollary) *and of the generic curve of* $V_{n,d}$. In fact, we prove our results for the subvariety $V'_{n,d} \subseteq V_{n,d}$ of those nodal curves which can be degenerated into a sum of n lines in general position ($V'_{n,d}$ is irreducible by [**5**, §11]). We then apply a recent result of Harris to the effect that $V_{n,d} = V'_{n,d}$ [**3**].

2. Zero-dimensional schemes. Let Hilb^e be the Hilbert scheme of zero-dimensional subschemes of degree e in $\mathbf{P}^2$. One can stratify Hilb^e: $Y, Z \in \mathrm{Hilb}^e$ belong to the same stratum iff $h^0(\mathbf{P}^2, I_Y(l)) = h^0(\mathbf{P}^2, I_Z(l))$ for all l. Let D^e denote the dense stratum. It is easy to show that D^e consists of m-regular (in the sense of Castelnuovo) schemes not lying on curves of degree $m-2$, where $m = \min\{i \in \mathbf{Z} | e \leq i(i+1)/2\}$. We denote by $\mathring{D}^e$ the subset of D^e consisting of schemes of the form $\sum_{i=1}^e P_i$, where $P_i \neq P_j$ for $i \neq j$ and $\sum_{k=1}^d P_{i_k} \in D^d$ for every $\{i_1, \ldots, i_d\} \subseteq \{1, \ldots, e\}$.

3. Main results. We need four propositions; they are of independent interest.

PROPOSITION 1. *Let* $d \leq (n-1)(n-2)/2$. *If a reduced curve of degree* n *with* d *assigned singular points* $P_1, \ldots, P_d$ *is not a specialization of an irreducible curve with* d *assigned nodes, then* $\sum_{i=1}^d P_i \notin D^d$.

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To prove Proposition 1, we suppose $\sum_{i=1}^{d} P_i \in D^d$ and make a reduction to a curve consisting of two smooth components. We then derive a contradiction by the Cayley-Bacharach theorem.

Let now $L = L_1 + \cdots + L_n \in \mathbf{P}^2$ be a sum of n general lines. Set $\{P_1\} = L_1 \cap L_2$, $\{P_2, P_3\} = \{(L_1 + L_2) \cap L_3\} \setminus \{P_1\}$, $\{P_4, P_5, P_6\} = \{(L_1 + L_2 + L_3) \cap L_4\} \setminus \{P_1, P_2, P_3\}$, etc. Then $\{P_1, \ldots, P_d\} \in D^d$ for $d \le n(n-1)/2$. By Severi [**5**, §11], for $d \le (n-1)(n-2)/2$, L with the assigned nodes $P_1, \ldots, P_d$ is a specialization of a curve of $V'_{n,d}$, and we can prove

PROPOSITION 2. *The scheme consisting of d nodes of a general curve of $V'_{n,d}$ is a point of $\overset{\circ}{D}{}^d$.*

In Proposition 3 below, we estimate the dimensions of some families of nonreduced curves. Let $f(x, y, z) = \sum a_{ijk} x^i y^j z^k$ be the homogeneous polynomial of degree n with generic coefficients. We consider the following system of $3d$ equations in a's,

$$(*) \qquad f'_x(x_1, y_1, z_1) = 0, \quad f'_y(x_1, y_1, z_1) = 0, \quad \ldots, \quad f'_z(x_d, y_d, z_d) = 0,$$

where $(x_1 : y_1 : z_1; \ldots ; x_d : y_d : z_d) \in (\mathbf{P}^2)^d$. Let $M^d \subset (\mathbf{P}^2)^d$ be the closed subscheme where the system has nontrivial solutions. We have two natural maps $\mathrm{Hilb}^d \overset{\phi_d}{\to} \mathrm{Sym}^d(\mathbf{P}^2) \overset{\sigma_d}{\leftarrow} (\mathbf{P}^2)^d$.

PROPOSITION 3. *We assume $[n(n+3)/6] \le d \le (n-1)(n-2)/2$ and $(n, d) \ne (6, 9)$.*

(i) *Let $K \subset M^d$ be an irreducible component and $(Q_1; \ldots ; Q_d) \in K$ a general point. If $d \ge n(n+3)/6$ and $\sigma_d(K) \cap \phi_d(\overset{\circ}{D}{}^d) \ne \varnothing$, then a curve of degree n having singularities at $Q_1, \ldots, Q_d$ is an irreducible nodal curve with d nodes and* $\dim K = \dim V_{n,d}$. *If $d = [n(n+3)/6]$, then there exists an irreducible nodal curve with d nodes in general position.*

(ii) *Let $C \in V'_{n,d}$ be a general curve and $P_1, \ldots, P_d$ its nodes. If l is the degree of a nonreduced curve of minimal degree having singularities at $P_1, \ldots, P_d$, then $l > n$ unless* $(n, d) = (8, 14)$.

We also need a generalization of a theorem of Arbarello-Cornalba and Zariski (cf. [**6**, Theorem 2]).

PROPOSITION 4. *Let $\mathcal{A}$ be an irreducible analytic family of curves of degree n with d assigned singular points whose general curve, say B, is reduced and has q singular points $P_1, \ldots, P_e, \ldots, P_d, \ldots, P_q$ $(e \le d \le q)$. We asusme $P_1, \ldots, P_d$ are the assigned singularities, $P_1, \ldots, P_e$ are nodes, and $P_{e+1}, \ldots, P_d$ are not nodes. We also assume:*

(i) *there exists a curve C of degree n with singularities at $P_1, \ldots, P_d$, and C and B have no common components,*

(ii) $\dim \mathcal{A} \ge \dim V_{n,d} - \min\{d - e, n + 1\}$. *Then* $\dim \mathcal{A} = \dim V_{n,d} - d + e$, *$q = d$, and $P_{e+1}, \ldots, P_d$ are cusps. Furthermore, if B is irreducible, we can drop condition* (i) *and replace* (ii) *by the condition:*

$$\dim \mathcal{A} \ge \dim V_{n,d} - \min\{d - e, 3(n-1)\}.$$

THEOREM. (i) *If* $n(n+3)/6 \leq d \leq (n-1)(n-2)/2$ *and* $(n,d) \neq (6,9)$, *then* $p_d\colon V'_{n,d} \to \mathrm{Sym}^d(\mathbf{P}^2)$ *is a birational morphism of* $V'_{n,d}$ *onto its image.*

(ii) *If* $d \leq \min\{n(n+3)/6, (n-1)(n-2)/2\}$ *and* $(n,d) \neq (6,9)$, *then for general* $P_1, \ldots, P_d \in \mathbf{P}^2$, *there exists a curve in* $V'_{n,d}$ *having nodes at* $P_1, \ldots, P_d$.

We prove both assertions simultaneously, first assuming that $[n(n+3)/6] \leq d \leq (n-1)(n-2)/2$ and $n \geq 7$. Choose an irreducible component $K \subset M^d$ such that $\sigma_d^{-1}(\overline{p_d(V'_{n,d})}) \subseteq K$ and $\dim K = \min\{\dim V'_{n,d}, 2d\}$. For this K, one can find a complete irreducible system W of nodal curves of degree n with d nodes such that $\overline{p_d(W)} = \sigma_d(K)$. We then show that $\dim W \cap V'_{n,d} \geq \dim V'_{n,d} - n - 1$, and this allows us to deduce from Proposition 4 that $V'_{n,d} = W$. The theorem then follows.

We observe that Part (ii) of Theorem also follows from [**1**, Theorem 3.2] together with [**3**]. From now on we assume $n(n+3)/6 \leq d \leq (n-1)(n-2)/2$ and $(n,d) \neq (6,9)$. Combining our results with the theorem of Harris [**3**] ($V_{n,d} = V'_{n,d}$), we obtain

COROLLARY. $\overline{p_d(V_{n,d})} = \overline{\sigma_d(M^d) \cap \phi_d(\mathring{D}^d)}$, *and for* $n(n+3)/6 \leq t \leq d$, *any* t *nodes of a general curve* $C \in V_{n,d}$ *determine the location of the remaining nodes of* C.

REMARK. We can solve (*) on the open subset of $M^d \cap \sigma_d^{-1}(\phi_d(\mathring{D}^d))$ where the system has unique solutions, and we obtain the equation of the generic curve of $V_{n,d}$.

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