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PROPERTIES OF PROPERLY EMBEDDED MINIMAL SURFACES OF FINITE TOPOLOGY

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Until the recent discovery of a sequence of properly embedded minimal surfaces with finite topology (Hoffman [**4**, **5**]; Hoffman and Meeks [**6**, **7**]), the only known examples were the plane, the catenoid and the helicoid. The existence of these new examples, which we will call M_k, $k \geq 1$, and others we have found (Callahan, Hoffman and Meeks [**1**]) makes it natural to ask qualitative questions about their behavior.

It is a fundamental fact, due to Osserman [**12**], that if a complete minimal surface has finite total curvature, then it is, conformally, a closed Riemann surface punctured in a finite number of points; finite total curvature implies finite topology. The minimal surface M_k has total curvature equal to $-4\pi(k+2)$. It is conformally a closed surface of genus k punctured in three points.

A natural question to ask is whether or not each of the surfaces M_k lies in a one-parameter family of complete embedded minimal surfaces of finite total curvature. It is known that the plane and the catenoid are the *unique* embedded examples of finite total curvature with their respective topologies. In particular they cannot be perturbed through embedded examples. However

THEOREM 1. *The surfaces M_k each lie in a smooth one-parameter family of embedded minimal surfaces.*

In the case of genus $k = 1$, the surface M_1 is conformally the square torus $\mathbf{C}/\mathbf{Z} \times \mathbf{Z}$, punctured in the three half-lattice points. It can be deformed through a family of embedded minimal surfaces which are, conformally, rectangular tori punctured in the three half-lattice points. In general, the surfaces M_k have two catenoid-type ends and one flat end. (By an *end* of a complete minimal surface of finite total curvature, we mean the image in $\mathbf{R}^3$ of a neighborhood of a puncture point. An embedded end is a "catenoid-type end" if it converges at infinity to an end of the catenoid. It is called a "flat end" if it converges at infinity to a plane. For embedded ends on complete minimal surfaces of finite total curvature, these are the only possibilities.) They contain $k+1$ straight lines which meet at a common point P and diverge into the flat end. The perturbations have three catenoid-type ends and contain no lines. The symmetry group of M_k is the dihedral group $D(2k+2)$, generated

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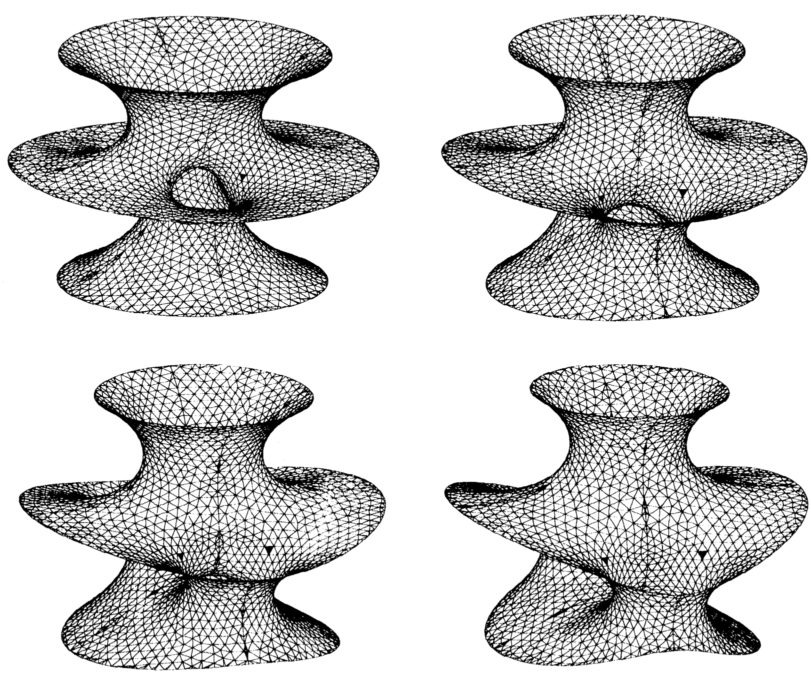


PERTURBED GENUS ONE EXAMPLES

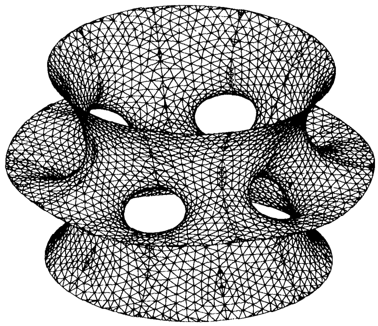
by rotation by π about one of the lines and a reflection in a plane that passes through P and bisects the angle made by two successive lines. This group has $4k+4$ elements. The symmetry group of a perturbation of M_k is generated by reflections through $k+1$ vertical planes and has $2k+2$ elements.

That the perturbations of M_k have a smaller symmetry group than M_k itself is an illustration of the fact that the surfaces M_k have maximal symmetry.

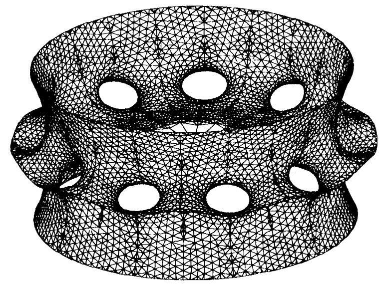
THEOREM 2. *Suppose N is a complete embedded minimal surface of finite total curvature, with genus k and three ends. If the symmetry group of N contains at least $4k+4$ elements then, up to homothety and rotation, N is equal to M_k.*

The construction of the examples M_k was greatly aided by the discovery that symmetries of the Gauss mapping of complete embedded minimal surfaces of finite total curvature correspond to symmetries of the surface. To be specific, let $\mathrm{Sym}(M)$ be the group of symmetries of M. Let $\mathrm{Iso}(M)$ be the group of intrinsic isometries of M. Finally, let $L(M)$ be the group of conformal diffeomorphisms h of M which have the following property: there exists an orthogonal motion A of $\mathbf{R}^3$ such that h is the lift to M of A via the Gauss map G;

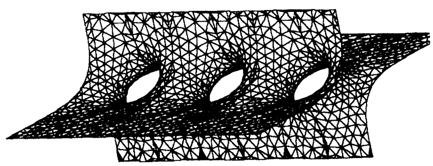
$$G \circ h = A \circ G.$$



GENUS FOUR EXAMPLE



GENUS NINE EXAMPLE



SCHERK'S SECOND SURFACE

THEOREM 3. *If M is a complete embedded minimal surface of finite total curvature, then*

$$L(M) = \mathrm{Iso}(M) = \mathrm{Sym}(M).$$

Our computer-generated pictures of the examples M_k suggested that as k grew large, the surfaces began to resemble a catenoid cut by a flat plane at the waist, with the circle of intersection replaced by a series of tunnels. In fact this can be made mathematically rigorous.

THEOREM 4. *The surfaces M_k converge, as a point set, to the union of a plane and the catenoid.*

All the interesting geometry in the above convergence procedure is happening near the limiting waist-circle of intersection, where the total Gaussian curvature on the surfaces M_k is concentrated and is blowing up as k goes to infinity. To see what is actually happening in this singular convergence, let N_k be the surface produced as follows. First homothetically expand the surface M_k by a factor which makes the maximum absolute value of the Gauss curvature equal to one. Then translate the surface so that a point of extreme curvature occurs at the origin.

THEOREM 5. *A subsequence of the surfaces N_k converges, up to similarity transformation, to Scherk's Second Surface:*

$$\sinh(x)\sinh(y) = \sin(z).$$

If the points of maximal curvature on each M_k are chosen carefully, the entire sequence will converge.

Each end of a complete immersed minimal surface M of finite total curvature is a properly immersed, once-punctured, compact disk with a well-defined limiting normal at the puncture point. It was shown in [**2**] that the image of a sufficiently small circle about the puncture point has a well-defined linking number with the line through the origin in the direction of the limiting normal. Define $n(M)$ to be the sum of all these linking numbers over all the ends of M. The total curvature of M can be written as

$$C(M) = 2\pi(\chi(M) - n(M)) = 2\pi(2 - 2k - r - n(M)),$$

where $\chi(M)$ is the Euler characteristic of M, r is the number of ends, and k is the genus of M.

THEOREM 6. *A complete connected minimal surface M of finite total curvature cannot have a point of self-intersection with multiplicity greater than $n(M) - 1$. If $n(M) = 2$, M is the catenoid. Also, $n(M)$ and r are either both even or both odd, even when M is nonorientable.*

The first statement of this theorem was also proved independently by R. Kusner [**11**]. The second statement follows easily from the first using Schoen's characterization of the catenoid [**14**]. The third statement has a purely topological proof.

The techniques used in constructing the examples M_k use the special properties of finite total curvature surfaces in an essential manner. At the present time, the only known example of a complete embedded minimal surface of finite topology with infinite total curvature is the helicoid. The following theorems can be seen as a beginning to the development of methods for dealing with properly embedded minimal surfaces of finite topology and infinite total curvature.

By an *annular end* of a surface, we mean an end that is homeomorphic to a compact disk punctured at one point. If the surface has a conformal structure, each annular end is conformally equivalent to either a punctured disk or a disk from which a smaller closed disk has been removed.

THEOREM 7. *A properly embedded minimal surface can have at most two pairwise disjoint annular ends with infinite total curvature.*

In particular, since an end with finite total curvature is conformally a punctured disk, no properly embedded minimal surface can have more than two annular ends which are conformally disks minus closed disks. Since a surface of finite topology can have only annular ends, it follows that a properly embedded minimal surface of finite topology is conformally a closed Riemann surface, punctured in a finite number of points, from which two or fewer disks have been removed.

We conjecture that:

1. An annular end of a properly immersed minimal surface must be conformally a punctured disk.

2. On a properly embedded minimal surface with more than one end, *every* annular end has finite total curvature. As the example of the helicoid shows, the hypothesis of more than one end is necessary.

The final theorem does not require the hypothesis of embeddedness.

THEOREM 8 (THE STRONG HALF-SPACE THEOREM). *If two properly immersed minimal surfaces are disjoint, then they are parallel planes. In particular, a properly immersed nonplanar minimal surface cannot lie in a half-space.*

This theorem fails without the hypothesis of properness, as demonstrated by the examples of Jorge and Xavier [**3**] and Rosenberg and Toubiana [**13**], which are complete minimal surfaces lying between two parallel planes in $\mathbf{R}^3$. (The Jorge-Xavier example has one annular end and the Rosenberg-Toubiana example has two annular ends. Each end in their examples is conformally a disk minus a disk. This illustrates that Conjecture 1 above requires the hypothesis of properness.) Also the result does not generalize to minimal hypersurfaces in $\mathbf{R}^n$. For example, the three-dimensional catenoid in $\mathbf{R}^4$ lies between two parallel planes.

These results will appear in [**8**, **9**, and **10**].

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REFERENCES

1. M. Callahan, D. Hoffman and W. H. Meeks III, *Embedded minimal surfaces with* 4 *ends*, preprint.

2. L. Jorge and W. H. Meeks III, *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology **22** (1983), 203–221.

3. L. Jorge and F. Xavier, *A complete minimal surface in* $\mathbf{R}^3$ *between parallel planes*, Ann. of Math. (2) **112** (1983), 203–206.

4. D. Hoffman, *The construction of families of embedded minimal surfaces*, to appear in the Proceedings of the Stanford Conference on Variational Methods for Free Surface Interfaces, September 1985.

5. ——, *The computer-aided discovery of new embedded minimal surfaces*, Math. Intelligencer (to appear).

6. D. Hoffman and W. H. Meeks III, *Complete embedded minimal surfaces of finite total curvature*, Bull. Amer. Math. Soc. **12** (1985), 134–136.

7. ——, *A complete embedded minimal surface in* $\mathbf{R}^3$ *with genus one and three ends*, J. Differential Geom. **21** (1985), 109–127.

8. ——, *The global theory of embedded minimal surfaces* (preprint).

9. ——, *One-parameter families of embedded minimal surfaces* (in preparation).

10. ——, *Limits of minimal surfaces and Scherk's second surface* (in preparation).

11. R. Kusner, *Conformal geometry and complete minimal surfaces*, Bull. Amer. Math. Soc. **17** (1987), 291–295.

12. R. Osserman, *A survey of minimal surfaces*, (2nd ed.), Dover Publications, New York (1986); or *Global properties of minimal surfaces in* E^3 *and* E^n, Ann. of Math. (2) **80** (1964), 340–364.

13. H. Rosenberg and E. Toubiana, *A cylindrical type complete minimal surface in a slab of* $\mathbf{R}^3$, preprint.

14. R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), 791–809.

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