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ALGEBRAIC VECTOR BUNDLES OVER REAL ALGEBRAIC VARIETIES

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By an affine algebraic variety, we mean in this note a locally ringed space $(X, \mathcal{R}_X)$ which is isomorphic to a ringed space of the form $(V, \mathcal{R}_V)$, where V is a Zariski closed subset in $\mathbf{R}^n$ and $\mathcal{R}_V$ is the sheaf of rings of regular functions on V. Recall that $\mathcal{R}_V(V)$ is the localization of the ring of polynomial functions on V with respect to the multiplicatively closed subset consisting of functions vanishing nowhere on V [**2**, **15**].

Let $\mathbf{F}$ be one of the fields $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ (quaternions). A continuous $\mathbf{F}$-vector bundle ξ over X is said to admit an algebraic structure if there exists a finitely generated projective module P over the ring $\mathcal{R}_X(X) \otimes_{\mathbf{R}} \mathbf{F}$ such that the $\mathbf{F}$-vector bundle over X, associated with P in the standard way, is C^0 isomorphic to ξ.

Our purpose is to study the following

PROBLEM. Characterize continuous $\mathbf{F}$-vector bundles over X which admit an algebraic structure.

This is an old problem, but despite considerable effort, the situation is well understood only in a few special cases: when X is the unit sphere S^n [**4**, **16**], when X is the real projective space $\mathbf{R}P^n$ [**5**, **7**] and when $\dim X \leq 3$ and $\mathbf{F} = \mathbf{R}$ [**8**, **9**] (cf. also [**13**] for a short survey).

Clearly, $\mathbf{R}P^n$ with its natural structure of an abstract real algebraic variety is actually an affine variety and every affine real algebraic variety admits a locally closed embedding in some $\mathbf{R}P^n$.

Let us first consider $\mathbf{C}$-vector bundles.

Let X be an affine nonsingular real algebraic variety and assume for a moment that X is embedded in $\mathbf{R}P^n$ as a locally closed subvariety. Consider $\mathbf{R}P^n$ as a subset of the complex projective space $\mathbf{C}P^n$. Let U be a Zariski neighborhood of X in the set of nonsingular points of the Zariski (complex) closure of X in $\mathbf{C}P^n$. Denote by $H^{\text{even}}_{\text{alg}}(U, \mathbf{Z})$ the subgroup of the cohomology group $H^{\text{even}}(U, \mathbf{Z})$ generated by the cohomology classes which are Poincaré dual to the homology classes in the Borel-Moore homology group $H_{\text{even}}(U, \mathbf{Z})$ represented by the closed irreducible complex algebraic subvarieties of U (cf. [**3**]). Let $H^{\text{even}}_{\mathbf{C}-\text{alg}}(X, \mathbf{Z})$ be the image of $H^{\text{even}}_{\text{alg}}(U, \mathbf{Z})$ via the restriction homomorphism $H^{\text{even}}(U, \mathbf{Z}) \to H^{\text{even}}(X, \mathbf{Z})$. One easily checks that $H^{\text{even}}_{\mathbf{C}-\text{alg}}(X, \mathbf{Z})$ does not depend on the choice of U or the choice of the embedding of X in $\mathbf{R}P^n$.

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THEOREM 1. *Let X be an affine nonsingular real algebraic variety and let ξ be a continuous* $\mathbf{C}$*-vector bundle over X. If ξ admits an algebraic structure, then the total Chern class $c(\xi)$ of ξ belongs to $H^{\mathrm{even}}_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$. Conversely, ξ admits an algebraic structure, provided that $c(\xi)$ belongs to $H^{\mathrm{even}}_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$, X is compact,* $\dim X \leq 5$ *and ξ is of constant rank.*

SKETCH OF PROOF. We can assume that X is a locally closed subvariety in $\mathbf{R}P^n$. Suppose that ξ admits an algebraic structure. Then one can find a Zariski neighborhood $\tilde{U}$ of X in the Zariski closure of X in $\mathbf{C}P^n$ and an algebraic vector bundle $\tilde{\xi}$ over $\tilde{U}$ such that the restriction $\tilde{\xi} \mid X$ of $\tilde{\xi}$ to X is C^0 isomorphic to ξ. It easily follows from [**3**] that $c(\xi)$ belongs to $H^{\mathrm{even}}_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$.

If all assumptions of the second part of Theorem 1 are satisfied, then with the help of the Grothendieck formula (cf. [**6**, p. 151]), one constructs a continuous $\mathbf{C}$-vector bundle η over X such that rank $\eta = 2$, η admits an algebraic structure and $c(\eta) = c(\xi)$ (here both assumptions, $c(\xi) \in H^{\mathrm{even}}_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$ and $\dim X \leq 5$ are essential). Since ξ is of constant rank, ξ and η are stably equivalent [**12**]. The conclusion follows now from [**16**, Theorem 2.2].

Our next step is the calculation of the groups $H^{2k}_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$ for a large class of varieties.

THEOREM 2. *Let X be a locally closed nonsingular algebraic subvariety of $\mathbf{R}P^n$ and let $X_{\mathbf{C}}$ be the Zariski closure of X in $\mathbf{C}P^n$. Assume that $X_{\mathbf{C}}$ is nonsingular. Then $H^{2i}_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$ is equal to the image of the restriction homomorphism*

$$H^{2i}(\mathbf{R}P^n, \mathbf{Z}) \to H^{2i}(X, \mathbf{Z})$$

in each of the following two cases:

(a) $2i \leq 2\dim X - n$.

(b) *$X_{\mathbf{C}}$ is an ideal theoretic complete intersection in $\mathbf{C}P^n$ and $2i < \dim X$.*

SKETCH OF PROOF. Consider the commutative diagram

$$\begin{array}{ccc} H^{2i}(\mathbf{C}P^n, \mathbf{Z}) & \xrightarrow{\gamma} & H^{2i}(X_{\mathbf{C}}, \mathbf{Z}) \\ {\scriptstyle\delta}\downarrow & & {\scriptstyle\beta}\downarrow \\ H^{2i}(\mathbf{R}P^n, \mathbf{Z}) & \xrightarrow{\alpha} & H^{2i}(X, \mathbf{Z}) \end{array}$$

where all homomorphisms are the restriction homomorphisms. If γ is an isomorphism, then $H^{2i}(X_{\mathbf{C}}, \mathbf{Z}) = H^{2i}_{\mathrm{alg}}(X_{\mathbf{C}}, \mathbf{Z})$ and β maps $H^{2i}(X_{\mathbf{C}}, \mathbf{Z})$ onto $H^{2i}_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$. Moreover, since δ is an epimorphism, $H^{2i}_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$ is equal to the image of α.

If (a) is satisfied, then γ is an isomorphism by the Lefschetz theorem [**1**].

If (b) is satisfied, then γ is an isomorphism by the Larsen theorem [**10**].

Notice that if (b) is satisfied and $\dim X$ is odd, then $H^{\mathrm{even}}_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$ is completely determined. For even $\dim X$, the situation is more complicated. Indeed, let

$$V^n = \{[x_0, \dots, x_n, x_{n+1}] \in \mathbf{R}P^{n+1} \mid x_0^2 + \cdots + x_n^2 = x_{n+1}^2\}.$$

Then the Zariski closure of V^n in $\mathbf{C}P^{n+1}$ is nonsingular and the restriction homomorphism $H^{\mathrm{even}}(\mathbf{R}P^{n+1}, \mathbf{Z}) \to H^{\mathrm{even}}(V^n, \mathbf{Z})$ is the zero homomorphism.

On the other hand, V^n is algebraically isomorphic to S^n and hence every continuous **C**-vector bundle over V^n admits an algebraic structure [**4**, **16**]. It follows from Theorem 1 that $H^n_{\mathbf{C}-\mathrm{alg}}(V^n, \mathbf{Z})$ is nontrivial, provided that n is even.

The example above indicates that the case in which $\dim X$ is even can only be handled under some additional assumptions.

Denote by $P(n;k)$ the projective space associated with the vector space of all homogeneous polynomials in $\mathbf{R}[x_0, \ldots, x_n]$ of degree k. If an element H in $P(n;k)$ is represented by a polynomial G, then $V(H)$ will denote the subvariety of $\mathbf{R}P^n$ defined by G.

THEOREM 3. *Let Y be a locally closed algebraic subvariety of $\mathbf{R}P^n$, $\dim Y \geq 2$. Assume that the Zariski closure of Y in $\mathbf{C}P^n$ is a nonsingular ideal theoretic complete intersection. Then there exists a nonnegative integer k_0 such that, for every integer k greater than k_0, one can find a subset Σ_k of $P(n;k)$ which is a countable union of proper subvarieties of $P(n;k)$ and has the property that for every H in $P(n;k)\backslash\Sigma_k$, $V(H)$ is either empty or nonsingular and transverse to Y and the group $H^{\mathrm{even}}_{\mathbf{C}-\mathrm{alg}}(Y \cap V(H), \mathbf{Z})$ is equal to the image of the restriction homomorphism*

$$H^{\mathrm{even}}(\mathbf{R}P^n, \mathbf{Z}) \to H^{\mathrm{even}}(Y \cap V(H), \mathbf{Z}).$$

In particular, if $Y = \mathbf{R}P^n$, then Theorem 3 determines $H^{\mathrm{even}}_{\mathbf{C}-\mathrm{alg}}$ for generic algebraic hypersurfaces in $\mathbf{R}P^n$, $n \geq 2$, of sufficiently high degree.

The proof of Theorem 3 is technically more complicated. Besides the Lefschetz theorem Moishezon's result [**11**, Theorem 5.4] also plays an essential role.

Theorems 2 and 3 show that, in many cases, Theorem 1 imposes severe restrictions on continuous **C**-vector bundles admitting an algebraic structure.

Among several applications of Theorem 3, we want to select only the simplest one.

THEOREM 4. *Let n be a positive integer. Then there exists a C^∞ embedding $h\colon S^n \to \mathbf{R}^{n+1}$, arbitrarily close in the C^∞ topology to the inclusion map, and a closed nonsingular algebraic subvariety X in $\mathbf{R}^{n+1}$ such that $h(S^n) = X$ and every continuous **C**-vector bundle over X admitting an algebraic structure is stably trivial. If $n = 4 \pmod 8$, then also every continuous **R**- or **H**-vector bundle over X admitting an algebraic structure is stably trivial.*

Theorem 4 is interesting in view of the fact that every continuous **F**-vector bundle over S^n admits an algebraic structure [**4**, **16**]. Let us also mention that every continuous stably trivial **F**-vector bundle admits an algebraic structure [**16**, Theorem 2.2].

The second part of Theorem 4 immediately implies that Shiota's conjecture [**14**, p. 1007] is false over X. Shiota has conjectured that a continuous **R**-vector bundle ξ of constant rank over an affine nonsingular compact real algebraic variety Y admits an algebraic structure if and only if all Stiefel-Whitney classes of ξ are Poincaré dual to the $\mathbf{Z}/2\mathbf{Z}$-homology classes of Y represented by closed algebraic subvarieties of Y. He proved the "only if" part of the conjecture and the "if" part is established in [**8**, **9**] for vector bundles over surfaces and threefolds.

SKETCH OF THE PROOF OF THEOREM 4. Let G be an element in $P(n+1; 2k+2)$ represented by the homogeneous polynomial

$$(x_0^2 + \cdots + x_n^2 - x_{n+1}^2)(x_0^2 + \cdots + x_n^2 + x_{n+1}^2)^k.$$

If we identify $\mathbf{R}^{n+1}$ with a subset of $\mathbf{R}P^{n+1}$ via the map which sends $(x_0, \ldots, x_n)$ to $[x_0, \ldots, x_n, 1]$, then $S^n = V(G)$. By Theorem 3 (applied to $Y = \mathbf{R}P^{n+1}$ and k sufficiently large) together with Theorem 1, there exists an element H in $P(n+1; 2k+2)$ such that H is arbitrarily close to G and for every continuous $\mathbf{C}$-vector bundle ξ over $X = V(H)$, the total Chern class of ξ is equal to 0. Clearly, there exists a C^∞ embedding $h\colon S^n \to \mathbf{R}^{n+1}$ which is close to the inclusion map and satisfies $h(S^n) = X$. Since X is diffeomorphic to S^n, the vector bundle ξ is stably trivial.

The second part of Theorem 4 follows by considering the complexification and the realification of vector bundles and by using the fact that the reduced Grothendieck group of continuous $\mathbf{F}$-vector bundles over X is isomorphic to $\mathbf{Z}$.

REFERENCES

1. A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. **69** (2) (1959), 713–717.
2. J. Bochnak, M. Coste and M. F. Roy, *Géométrie algébrique réelle*, Ergeb. Math. Grenzgeb., Vol. 12, Springer-Verlag, New York, 1987.
3. A. Borel and A. Haefliger, *La classe d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France **89** (1961), 461–513.
4. R. Fossum, *Vector bundles over spheres are algebraic*, Invent. Math. **8** (1969), 222–225.
5. A. V. Geramita and L. G. Roberts, *Algebraic vector bundles on projective spaces*, Invent. Math. **10** (1970), 298–304.
6. A. Grothendieck, *La théorie des classes de Chern*, Bull. Soc. Math. France **86** (1958), 137-154.
7. J. P. Jouanolou, *Comparison des K-theories algébrique et topologique de quelque variétés algébrique*, C. R. Acad. Sci. Paris Ser. A **272** (1971), 1373–1375.
8. W. Kucharz, *Vector bundles over real algebraic surfaces and threefolds*, Compositio Math. **60** (1986), 209–225.
9. ——, *Topology of real algebraic threefolds*, Duke Math. J. **53** (1986), 1073–1079.
10. M. E. Larsen, *On the topology of complex projective manifolds*, Invent. Math. **19** (1973), 251–260.
11. B. G. Moishezon, *Algebraic homology classes on algebraic varieties*, Math. USSR-Izv. **1** (1967), 209–251.
12. F. P. Peterson, *Some remarks on Chern classes*, Ann. of Math. (2) **60** (1959), 414–420.
13. L. G. Roberts, *Comparison of algebraic and topological K-theory*, Algebraic K-Theory II, Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin and New York, 1973, pp. 74–78.
14. M. Shiota, *Real algebraic realization of characteristic classes*, R. I. M. S. Kyoto Univ. **18** (1982), 995–1008.
15. R. Silhol, *Géométrie algébrique sur un corps non algébriquement clos*, Comm. Alg. **6** (1978), 1131–1155.
16. R. G. Swan, *Topological examples of projective modules*, Trans. Amer. Math. Soc. **230** (1977), 201–234.

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