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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 1, July 1987

0273-0979/87 $1.00 + $.25 per page

Nonlinear approximation theory, by Dietrich Braess, Springer Series in Computational Mathematics, vol. 7, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1986, xiv + 290 pp., $69.50. ISBN 0-387-13625-8

Approximation theory arose out of the need to represent "difficult" functions by "simpler" functions, precision being then traded for ease of computation. The theory concerns itself more with *classes* of functions than with individual functions. A central problem of perennial interest starts with a prescribed set M in a normed space E. One contemplates the approximation of an element f in E by an element of M. The least error possible in this process is $d(f, M)$, defined to be the infimum of $\|f - m\|$ as m ranges over M. If an element m has the property $\|f - m\| = d(f, M)$, it is called a "best approximant" or "nearest point." Basic questions then are whether a nearest point exists, and if it does, whether it is unique, how it is to be recognized, and how it is to be computed. When a sequence of subsets M_n is given, interesting *asymptotic* questions arise, such as whether the sequence $d(f, M_n)$ converges to zero and if so how rapidly.

If the subset M is a linear subspace of E, the approximation problem is classified as linear, notwithstanding the fact that nearest points usually depend upon the element being approximated in a *nonlinear* manner. All other cases are, of course, *nonlinear*. If each element of E has at least one nearest point in M, the latter is termed an *existence set*. If there is always exactly one nearest point, M is a *Chebyshev set*. An existence set M is a *sun* if each f not in M has a nearest point m such that m is also a nearest point to each element on the ray issuing from m through f. Many other special properties have been observed and studied, and one branch of approximation theory seeks to determine which sets in which normed spaces have one or the other of these properties. As an example of what is known, there is the theorem of Sz.-Nagy (1942) asserting that each closed and convex set in a uniformly convex Banach space is a Chebyshev set. As an example of what is not known, there is the question, posed by Klee (1961), of whether each Chebyshev set in Hilbert space is convex.

Topics such as the foregoing belong to the abstract side of the subject. There is also a practical side, in which specific spaces E and subsets M are studied because of their usefulness in other branches of mathematics or in applications. For example, E is often a space of continuous functions on a compact interval, and M might be a set of rational functions of specified degree, or trigonometric sums having specified length but variable frequencies. Piecewise polynomials ("spline functions") with variable knots or linear families of functions subjected to constraints are other useful examples of the nonlinear theory.

The book under review is a thorough treatise on nonlinear approximation and is the only general book on this subject to have appeared. It sets a standard for clarity and completeness that will be difficult to surpass. The author has long been a principal contributor to the subject and brings to this work unique gifts for combining and unifying diverse results that hitherto were scattered throughout the periodical literature. One effect of this work is to standardize the nomenclature, and another is to bring special theories under single general theories where possible.

The book has two main parts. The first, devoted to the general abstract theory, has four chapters. Chapter I reviews the linear theory and covers such topics as Kolmogorov's theorem and Haar's theorem, both of which concern best approximation in spaces of continuous functions. Chapter II develops nonlinear approximation using tools from functional analysis. Here one finds general theorems on existence, unicity, and characterization of nearest points. Chapter III is devoted to results obtained by methods of local analysis. The concepts of tangent cones and critical points play a leading role. Chapter IV exploits methods of global analysis, culminating in a unicity theorem due to Wulbert (1971) and Braess (1973) for nearest points on Haar manifolds. The remaining four chapters develop the theory of four concrete types of nonlinear approximation: rational functions, exponential sums, spline functions with free knots, and γ-polynomials. These four chapters average 40 pages each and provide an invaluable summary of the current status of these topics.

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