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REPRESENTATIONS OF MOD p LIE ALGEBRAS

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Let G be a semisimple, simply connected algebraic group defined over an algebraically closed field k of characteristic $p > 0$. Because any rational G-module inherits the structure of a restricted module (in the sense of [**4**, p. 188]) for the Lie algebra $\mathcal{G}$ of G, the representation theory of $\mathcal{G}$ has primarily focused on the study of restricted modules. We outline here our recent investigations of the more general—that is, not necessarily restricted—represention theory of $\mathcal{G}$. Details will appear in [**3**].

We approach the representation theory of $\mathcal{G}$ through that of a family of finite-dimensional quotient algebras of the universal enveloping algebra $U(\mathcal{G})$ of $\mathcal{G}$. As described below, these algebras are parametrized by characters on a certain abelian subalgebra $\mathcal{O}$ of $U(\mathcal{G})$. Because the restricted enveloping algebra $V(\mathcal{G})$ appears as a distinguished member of this family (the others being thought of as "deformations" of $V(\mathcal{G})$), a better understanding of the representation theory of these algebras may lead to a clearer picture of that of $V(\mathcal{G})$.

For a restricted Lie algebra $\mathcal{G}$, we employ the central subalgebra $\mathcal{O} \subset U(\mathcal{G})$ considered by Zassenhaus in his foundational paper [**10**]. Namely, $\mathcal{O}$ is the image of the semilinear monomorphism $S^{\cdot}(\mathcal{G}) \to U(\mathcal{G})$ defined on the symmetric algebra $S^{\cdot}(\mathcal{G})$ of $\mathcal{G}$ by sending $X \in \mathcal{G}$ to $X^p - X^{[p]}$. Using the Jordan decomposition of the dual $\mathcal{G}^*$ given in [**7**], we obtain properties such as "regular", "semisimple", or "nilpotent" for characters $\chi\colon \mathcal{O} \to k$ whenever $\mathcal{G} = \mathrm{Lie}(G)$.

PROPOSITION 1. *Let $\mathcal{G}$ be a restricted Lie algebra of dimension d and let $\chi\colon \mathcal{O} \to k$ be a character with associated one-dimensional $\mathcal{O}$-module k_χ. Then $A_\chi \equiv U(\mathcal{G}) \otimes_{\mathcal{O}} k_\chi$ is a Frobenius algebra of dimension p^d. Moreover, for each irreducible $\mathcal{G}$-module M, there is a unique character $\chi\colon \mathcal{O} \to k$ such that the action of $U(\mathcal{G})$ on M factors through A_χ.*

Of course, if $\chi = 0$ then $A_\chi \cong V(\mathcal{G})$, the restricted enveloping algebra of $\mathcal{G}$.

One easily proves that $\mathrm{Ext}^{\cdot}_{U(\mathcal{G})}(M, M') = 0$ whenever M is an A_χ-module, M' is an $A_{\chi'}$-module, and $\chi \neq \chi'$. On the other hand, nontrivial computations are facilitated by the following spectral sequence.

PROPOSITION 2. *Let $\mathcal{G}$ and A_χ be as in Proposition 1. If M and N are two A_χ-modules with M finite-dimensional, then there is a natural spectral*

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sequence

$$E_2^{s,t}(M,N) = \mathrm{Ext}^s_{A_\chi}(M,N) \otimes \Lambda^t \mathcal{G}^* \Rightarrow \mathrm{Ext}^{s+t}_{U(\mathcal{G})}(M,N).$$

As in our earlier work [**1**, **2**] concerning the cohomology of restricted representations, we define the cohomological support variety $|\mathcal{G}|_M$ of a finite-dimensional A_χ-module M to be the variety of the annihilator ideal in $H^{ev}(V(\mathcal{G}),k)$ of $\mathrm{id}_M \in \mathrm{Ext}^0_{A_\chi}(M,M)$. Most conclusions we obtain involve the image of $|\mathcal{G}|_M$ under the natural (finite) map $\varphi\colon \mathrm{Spec}(H^{ev}(V(\mathcal{G}),k)) \to \mathcal{G}$ (cf. [**1**, 2.1]). For example, M is a projective A_χ-module if and only if $\varphi(|\mathcal{G}|_M) = \{0\}$; $\dim|\mathcal{G}|_M = \dim\varphi(|\mathcal{G}|_M)$ equals the rate of growth of a minimal projective resolution of M as an A_χ-module.

Using the corresponding result for restricted representations [**2**, 1.4], we obtain the following representation-theoretic interpretation of $\varphi(|\mathcal{G}|_M)$.

THEOREM 3. *Let $\mathcal{G}$ be a finite-dimensional restricted Lie algebra, $\chi\colon \mathcal{O} \to k$ a character, and M a finite-dimensional A_χ-module. For any $X \in \mathcal{G}$, let $\{X\}$ denote the subalgebra of A_χ generated by X. Then a nonzero element $X \in \mathcal{G}$ lies in $\varphi(|\mathcal{G}|_M)$ if and only if both $X^{[p]} = 0$ and M is not a projective $\{X\}$-module.*

Clearly, $\varphi(|\mathcal{G}|_M)$ is a closed conical subvariety of $\mathcal{G}$. Using the above theorem, we can prove that whenever $Y \subset \mathcal{G}$ is a subvariety of $\mathcal{G}$ representable for some finite-dimensional A_χ-module M as $\varphi(|\mathcal{G}|_M)$, then any closed conical subvariety of Y has the form $\varphi(|\mathcal{G}|_N)$ for a finite-dimensional A_χ-module N.

We specialize to $\mathcal{G} = \mathrm{Lie}(G)$ for G a simply connected semisimple algebraic group over k of rank l (with chosen Borel subgroup $B = T \cdot U$ containing a maximal torus T and having unipotent radical U). Since any $\chi\colon \mathcal{O} \to k$ is G-conjugate to a character vanishing on $\mathcal{U} = \mathrm{Lie}(U)$, we restrict attention to such characters χ. For each of the p^l weights $\lambda\colon \mathcal{H} \equiv \mathrm{Lie}(T) \to k$ satisfying

$$\lambda(H^{[p]}) - \lambda(H)^p = \chi(H^{[p]} - H^p), \qquad H \in \mathcal{H},$$

we consider the "Verma-type" A_χ-module $V_{\chi,\lambda}$ of dimension $p^{\dim(\mathcal{U})}$ induced from the one-dimensional $\mathcal{B} = \mathrm{Lie}(B)$-module defined by λ.

For example, for $\mathcal{G} = \mathrm{sl}_2$ $(p \neq 2)$, there are three SL_2-orbit types of characters corresponding to χ regular semisimple, regular nilpotent, or 0. If χ is regular semisimple, then A_χ is a semisimple algebra. If χ is regular nilpotent, each $V_{\chi,\lambda}$ is irreducible but only one is projective, whereas the projective covers of the nonprojective $V_{\chi,\lambda}$ are self-extensions of $V_{\chi,\lambda}$ with two composition factors. If $\chi = 0$, then the $V_{\chi,\lambda}$ are indecomposable but only one is irreducible (and projective).

With the generous assistance of O. Gabber, the authors have formulated and proved the following theorem extending a basic theorem of Kac and Weisfeiler for irreducible A_χ-modules [**6**, Theorem 2]. If A is a k-algebra, let A-mod denote the category of left A-modules.

THEOREM 4. *Let G be a semisimple simply connected algebraic group with Lie algebra $\mathcal{G}$. Assume that p is good for the root system of G (cf. [**9**, p. 178]). Let $\chi\colon \mathcal{O} \to k$ be a character with Jordan decomposition $\chi = \chi_s + \chi_n$ and let*

$\mathfrak{z}$ be the centralizer in $\mathcal{G}$ of the semisimple character χ_s. Then $\mathfrak{z}$ is the Levi factor of a parabolic subalgebra $\mathcal{P} = \mathfrak{z} \oplus \mathcal{N}$. Consider the functors

$$\psi = (-)^N : A_\chi\text{-mod} \to Z\text{-mod},$$
$$\theta = A_\chi \otimes_P (-) : Z\text{-mod} \to A_\chi\text{-mod}$$

where Z (respectively, N, P) denotes the subalgebra of A_χ generated by $\mathfrak{z}$ (resp., $\mathcal{N}, \mathcal{P}$). Then θ is left adjoint to ψ, both θ and ψ are exact functors, and the adjunction morphisms $\mathrm{id}_{Z\text{-mod}} \to \psi \circ \theta$ and $\theta \circ \psi \to \mathrm{id}_{A_\chi\text{-mod}}$ are isomorphisms.

The preceding theorem relates the study of the representation theory of A_χ for an arbitrary character $\chi : \mathcal{O} \to k$ to that of the representation theory of a possibly smaller Lie algebra ($\mathfrak{z}' = [\mathfrak{z}, \mathfrak{z}]$ in the above theorem) and a nilpotent character (the restriction of χ_n to $\mathfrak{z}'$). For example, we readily obtain a strengthening of a result of Rudakov [**8**] concerning minimal projective A_χ-modules, as well as necessary and sufficient conditions for A_χ to be a semisimple algebra. In conjunction with Proposition 2, this yields an explicit calculation of $\mathrm{Ext}^{\bullet}_{U(\mathcal{G})}(M, N)$ for "most" irreducible $U(\mathcal{G})$-modules M, N.

In seeking to generalize the example of sl_2 discussed above for regular nilpotent characters $\chi : \mathcal{O} \to k$, the authors have investigated the suggestive discussion of Kac in [**5**]. We obtain the following theorem.

THEOREM 5. *Let G be a simply connected simple algebraic group of type A_l, B_l, C_l or G_2 and let $\mathcal{G} = \mathrm{Lie}(G)$. Assume that p does not divide the order of the quotient of the weight lattice by the root lattice. For any regular nilpotent character $\chi : \mathcal{O} \to k$, $V_{\chi,\lambda}$ is an irreducible A_χ-module.*

Using Theorem 5, we can obtain much useful information concerning the representation theory of A_χ for χ regular nilpotent. We employ the notation $\sim$ for the equivalence relation on the set X of integral weights $\lambda : \mathcal{H} \to k$ defined by $\lambda \sim \mu$ if and only if $w(\lambda + \rho) = \mu + \rho$ for some element w of the Weyl group W of G (where ρ is the half-sum of the positive roots).

THEOREM 6. *Let G and p be as in Theorem 5 and assume that χ is a regular nilpotent character. Let S_λ denote the irreducible A_χ-module $V_{\chi,\lambda}$, $\lambda \in X$.*

(a) *For $\lambda, \mu \in X$, $\mathrm{Ext}^{\bullet}_{A_\chi}(S_\lambda, S_\mu) \neq 0 \Leftrightarrow \lambda \sim \mu \Leftrightarrow S_\lambda \cong S_\mu$.*

(b) *The projective cover of S_λ as an A_χ-module has a composition series with $|\{\mu \in X : \lambda \sim \mu\}|$ factors, each isomorphic to S_λ.*

(c) *$S_{-\rho}$ is the unique (up to isomorphism) projective, irreducible A_χ-module.*

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