The book under review, which is the outgrowth of a series of introductory lectures, is written on two levels. The initial portion of each chapter deals with the shift operator of unit multiplicity and delves quite deeply into the associated function theory, paying special attention to the spectral analysis of the parts of the backward shift operator. This material could serve as an introduction to shift-related operator theory and function theory for someone with a basic background in functional analysis and complex analysis. All but a few chapters contain supplementary sections where the earlier material is refined and extended; in particular, multiple shift operators are studied. The style here becomes that of an advanced monograph. Besides the eleven main chapters there are five appendices, themselves comprising about 45 percent of the text. One, of 100 pages, gives an introduction to the spectral theory of two kinds of operators closely related to the shift operator, Hankel and Toeplitz operators; another, of 56 pages and contributed by S. V. Hruščev and V. V. Peller, further develops the theory of Hankel operators, especially the connections of those operators with approximation problems and with stationary Gaussian sequences.

This is a book for the devotee, or the would-be devotee. If my experience is typical, those who love the subject will love the book.

DONALD SARASON

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An introduction to nonstandard real analysis, by A. E. Hurd and P. A. Loeb, Pure and Applied Mathematics, vol. 118, Academic Press, 1985, xii + 232 pp., $35.00. ISBN 0-12-362440-1

Nonstandard analysis is now widely applied in a number of different mathematical fields. A partial list of the applications includes functional analysis (Bernstein and Robinson **[15]**, and the survey by Henson and Moore **[21]**), perturbation theory (Lutz and Goze **[38]**), mathematical physics (Arkeryd **[9, 10, 11, 12, 13]**), potential theory (Loeb **[37]**), mathematical economics (Brown and Robinson **[16, 17]**, Anderson **[4, 6, 8]**, the references in **[7]**, and the forthcoming book by Rashid **[46]**), and probability theory (see the survey by Cutland **[18]**).

Standard mathematicians tend to test the worth of nonstandard analysis by asking whether it has led to new standard results in their fields. It is not clear that this is the right test: after all, most fields yield far more results of internal interest than applications to other fields. Nonetheless, it is a test which nonstandard analysis is beginning to meet.

In most of the above areas, nonstandard analysis has led to new standard theorems. A metatheorem guarantees that any standard theorem provable by nonstandard methods has a standard proof; this is important, since it tells us that any theorem with a nonstandard proof follows from the usual axioms of analysis. However, there is no guarantee that the standard proof will be reasonable: it may be very long, or exceedingly unintuitive. In most cases, the nonstandard proofs have been followed by standard proofs; however, in a few cases, nonstandard analysis still provides the only known proofs for these results.

This book is one of three appearing roughly simultaneously: Stroyan and Bayod [**52**] has just appeared, and Albeverio et al. [**1**] is in press. What sets these three books apart from earlier texts in nonstandard analysis (Robinson [**47**], Stroyan and Luxemburg [**51**], Davis [**19**], and Lutz and Goze [**38**]) is the inclusion of the developments of the last decade in nonstandard formulations of measure theory, particularly the Loeb measure. The Loeb measure has had applications in several of the above areas, but has had its greatest impact in probability theory. For this reason, we shall concentrate particularly on probability theory in this survey. The reader interested in a more detailed survey should consult Cutland [**18**].

There are two dialects in nonstandard analysis. One (which we shall call NSA) has grown from Robinson's original formulation of the subject [**47**] and is the dialect which is presented by Hurd and Loeb. The second dialect, internal set theory (IST), was developed later by Edward Nelson [**39**], with probability theory and mathematical physics as Nelson's main intended applications. The principal advantage of internal set theory is that it is relatively easy to learn. In particular, the distinction between internal and external sets, which poses significant difficulties to the newcomer to NSA, is finessed in IST: external sets cannot be defined. As a consequence, Loeb measure (and other constructions such as the nonstandard hull) cannot be defined within IST, although certain paraphrases of Loeb measure arguments are possible. Thus, while IST is easier to learn than NSA, it is less flexible. Hurd and Loeb give a very good treatment of NSA, making the learning process as painless as possible.

The basic idea of the Loeb measure construction [**36**] is easy to describe. Suppose Ω is an internal *-finite set, i.e. an infinite set constructed in a nonstandard model which satisfies all the formal properties of finite sets. We suppose ν is normalized counting measure on Ω, i.e. $\nu(A) = |A|/|\Omega|$ for all (internal) $A \subset \Omega$. Any finite element r of the nonstandard real numbers is infinitely close to a unique standard real number, denoted ${}^\circ r$. $\nu(A)$ is a nonstandard real number between 0 and 1, and hence is finite. Let $\mu(A) = {}^\circ\nu(A)$. Thus, μ is a finitely additive measure in the ordinary standard sense; its domain $\mathscr{A}$, the class of all internal subsets of Ω, is an algebra. One can construct nonstandard models with the following saturation property: if A_n is a sequence of internal sets and $\bigcap_n A_n = \varnothing$, then there is a finite collection $\{n_1, \ldots, n_k\}$ such that $\bigcap_i A_{n_i} = \varnothing$. Thus, the countable additivity hypothesis of the Carathéodory extension theorem is trivially satisfied. Accordingly, there is a unique extension of μ to a countably additive measure on the σ-algebra generated by $\mathscr{A}$. The completion of this extension, denoted $\bar{\nu}$, is called the Loeb measure generated by ν. It should be emphasized that, although $\bar{\nu}$ is obtained from a nonstandard construction, it is a countably additive probability measure in the usual standard sense. Loeb originally developed this construction in the course of constructing a new standard ideal boundary for potential theory [**37**].

The Loeb measure construction is very simple, but it has had profound consequences. (Ω, ν) behaves just like an ordinary discrete set with a counting measure: integration is just summation. On the other hand, $(\Omega, \bar{\nu})$ is a probability space in the usual standard sense. A series of so-called lifting theorems (beginning with Loeb [**36**] and continuing in Anderson [**2**] and Keisler [**28**]) link the theories of integration and stochastic processes on the two spaces, providing a foundation for probability theory that simultaneously combines the features of discrete and continuous entities. The simplest example is Lebesgue measure. We take n to be an infinite element of the nonstandard natural numbers, and let $\Omega = \{0, 1/n, \ldots, (n-1)/n, 1\}$. Ω is $*$-finite. We let $h: \Omega \to [0,1]$ be defined by $h(x) = {}^\circ x$. A set $B \subset [0,1]$ is Lebesgue measurable iff $h^{-1}(B)$ is $\bar{\nu}$-measurable, in which case $\bar{\nu}(h^{-1}(B))$ is the Lebesgue measure of B (Anderson [**2, 5**]; the "if" part is due to Henson and Fisher).

A similar construction works for Brownian motion. Let $\Omega = \{1, -1\}^n$, where n is an infinite natural number. We can define a random walk by setting

$$\chi(t, \omega) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \omega_i.$$

For each $t \in [0,1]$, define $\beta(t, \omega) = {}^\circ\chi(t, \omega)$. Then β is a standard Brownian motion on the probability space $(\Omega, \bar{\nu})$ (Anderson [**2**]). Wiener measure and Donsker's invariance principle on the convergence of random walks to Wiener measure fall out immediately.

Lawler [**32**] used the relationship of the random walk χ to Brownian motion to study self-avoiding random walks, which are of interest in polymer chemistry and theoretical physics. His proofs are carried out in IST, so the following description is a restatement of his argument in NSA. Dvoretzky, Erdős, and Kakutani [**20**] showed that, if β is a Brownian motion in $\mathbf{R}^k$ with $k \geqslant 4$, then almost every path of β has no intersection with itself. Since ${}^\circ\chi$ (more precisely, its k-dimensional analogue) *is* a Brownian motion, almost every path has the property that all self-intersections are of infinitesimal length. Consequently, if one lets γ be the process obtained by erasing the loops in χ, then the paths of ${}^\circ\gamma$ are exactly the same as those of the original Brownian motion β, except that the process is speeded up. Lawler shows in [**32**] that, for $k \geqslant 5$, the speeding-up is by a finite factor which is uniform over time and paths. He has subsequently shown that, for $k = 4$, the speeding-up is by an infinite factor (involving $\log n$), uniform over paths. It is an immediate corollary that the distribution of a standard loop-erased random walk, speeded up appropriately, converges weakly to standard Brownian motion for $k \geqslant 4$ as $n \to \infty$. This is a new standard result, obtained first by nonstandard methods.

Stoll [**50**] has recently obtained results on a different formulation of self-avoiding random walks in $\mathbf{R}^2$. He considers the analogue of χ in two dimensions, but uses an internal probability measure ν' in which the probability of a path is decreased according to a potential function whenever the path comes close to intersecting itself; a special case of such a potential is one in which the probability of each path is proportional to e^{-cn}, where n is the number of self-intersections. He is able to characterize the distribution of $^\circ\chi$; in particular, it is absolutely continuous with respect to Wiener measure. As usual, a weak convergence result on finite random walks follows immediately; this result substantially generalizes the potentials that had been allowed in the previous standard literature.

Lévy Brownian motion is a generalization of the usual Brownian motion in which the time parameter t is taken in $\mathbf{R}^d$. Stoll [**49**] generalizes the above construction of Brownian motion to a construction of Lévy Brownian motion. As a consequence, he obtains a new standard invariance principle on the convergence of stochastic processes with time parameter defined on a lattice to Lévy Brownian motion as the lattice becomes finer.

The local time of a Brownian path $\beta(\cdot, \omega)$ is

$$l(t, x, \omega) = \frac{d}{dx}\int_0^t I(\beta(s, \omega) \leqslant x)\, dx,$$

where I is the indicator function. In a series of papers, Perkins has given a nonstandard construction of, and proven several new standard theorems on, local time. The nonstandard construction simply counts the number of times the random walk χ visits each possible lattice point, and normalizes appropriately [**40**]. A new standard invariance principle falls out immediately [**42**]. Perkins [**40**] also establishes the first standard characterizations of $l(t, x, \omega)$ which are intrinsic (i.e., depend only on $\{s \leqslant t\colon \beta(s, \omega) = x\}$) and global (i.e., there is a single null set Ω^0 so that the characterization works on the complement of Ω^0, uniformly in x). The key to obtaining a global characterization is the fact that one can compare the internal measures of internal sets of Loeb measure zero, and hence it is possible to show in some situations that the measure of an uncountable union of null sets is a null set. An exact characterization of the Hausdorff measure of the level sets of Brownian motion follows by standard arguments (Perkins [**41**]).

The stochastic integral with respect to a Brownian motion is motivated by the definition of a Stieltjes integral. The barrier to directly defining it as a Stieltjes integral is that the paths of a Brownian motion are almost surely not of bounded variation. However, the paths of the random walk χ have well-defined nonstandard variation $\sqrt{n}$. Hence, if $g\colon {}^*[0,1] \times \Omega \to {}^*\mathbf{R}$ is internal (where ${}^*\mathbf{R}$ denotes the nonstandard real numbers and ${}^*[0,1]$ those elements of ${}^*\mathbf{R}$ between 0 and 1), the Stieltjes integral $\int g\, d\chi$ is perfectly well defined; in fact, it is obtained as a *-finite summation. Anderson [**2**] shows that, if $f\colon [0,1] \times \Omega \to \mathbf{R}$ is Itô integrable with respect to β, then there is a lifting $g\colon {}^*[0,1] \times \Omega \to {}^*\mathbf{R}$ such that $^\circ\!\int g\, d\chi$ is the Itô integral $\int f\, d\beta$. This opens up the possibility of solving stochastic differential equations (with respect to Brownian motion as well as other stochastic processes) by solving a nonstandard stochastic difference equation. The existence of a solution to the difference equation is trivial; one then needs to show that the standard part of the solution to the difference equation solves the differential equation. This approach has been exploited extensively in Keisler's monograph [**28**]. Keisler obtains existence theorems for solutions of a more general class of stochastic differential equations: see in particular Theorem 5.14 of [**28**]. In addition, while standard approaches yield weak solutions (i.e., solutions which live on a larger probability space), Keisler shows that stochastic differential equations on Loeb spaces have strong solutions, living on the same Loeb space. He also gives theorems showing that stochastic processes with the same distributions which live on Loeb spaces can be mapped onto each other.

It is possible to give discrete nonstandard representations to more general stochastic processes, in particular semimartingales. Then Keisler's techniques can be used to study stochastic differential equations with respect to these more general processes (Hoover and Perkins [**23**, **24**] and Lindstrom [**33**, **34**, **35**]). Their existence theorem, which extended the previously known standard results, was completed at the same time as a standard proof by Jacod and Memin [**25**]; as in Keisler [**28**], the nonstandard approach yields strong solutions, while Jacod and Memin obtained weak solutions.

It had been known for some time that every nonnegative submartingale was equal in distribution to the absolute value of a martingale. Perkins [**43**] used the machinery in [**23**] to extend this to local submartingales and give an explicit formula for the local martingale that results in terms of the Doob-Meyer decomposition of the original local submartingale.

Perkins [**45**] has also obtained new results on Dawson's measure-valued diffusions. Specifically, he obtains more precise information on the Hausdorff dimension.

Since Loeb measures have also played an important role in the applications to mathematical economics, we shall give a very brief hint of the nature of a few of those applications. Aumann [**14**] introduced into mathematical economics the use of nonatomic measure spaces as models for the set of agents in "large" economies, in much the same way as measure spaces are used to model large discrete systems in mathematical physics. Aumann showed the equivalence, in his model, of two notions of equilibrium: Walrasian equilibrium (i.e., an equilibrium notion mediated by prices in which supply equals demand) and the core, an equilibrium notion arising from game theory which focuses on the ability of coalitions of agents to exploit market power. There ensued an extensive literature, involving many authors, deriving asymptotic results on the relationship of core and Walrasian equilibrium in large finite economies using Aumann's result and the theory of weak convergence of probability measures (in particular, Skorokhod's Theorem); for an account of this work, see Hildenbrand's book [**22**]. Brown and Robinson [**17**] introduced nonstandard exchange economies (i.e., economies in which there is a *-finite set of agents) and showed that the appropriate notions of core and Walrasian equilibrium coincided. Their work was done before the discovery of the Loeb measure; in hindsight, we see that the equilibrium notions used in Brown and Robinson [**17**] are essentially the liftings of Aumann's equilibrium notions on the Loeb measure economies generated by the *-finite economies. The advantage of the nonstandard approach is apparent in Brown and Robinson [**16**]; asymptotic theorems for large finite economies follow almost immediately from Brown and Robinson [**17**] by the transfer principle. Thus, nonstandard analysis obtained results comparable to those obtained by the measure-theoretic approach more or less simultaneously and with much less work; see Anderson **[7]** for a comparison of the results obtained by the two approaches. A translation process applied to the nonstandard proofs led to short, completely elementary (i.e. using neither measure theory nor nonstandard analysis) convergence proofs (Anderson **[3, 4]**).

There are two nonstandard economics results for which no tractable standard proof is yet known. An agent is said to have a convex preference if, given any possible consumption x, the set of consumptions the agent prefers is a convex set. The sense in which the core converges to the set of Walrasian equilibrium is stronger in general when preferences are convex than when they are not. However, Anderson **[6]** shows that, even if preferences are nonconvex, convergence holds in the strong sense with probability one. One can give a standard proof of this result in the important special case in which the limit economy of the sequence has a finite or countable number of Walrasian equilibrium prices, but no standard proof is known in the general case. A related result concerns Pareto optima (i.e., allocations of resources with the property that there is no reallocation of resources which makes every agent better off). A fundamental theorem of economics is that, with convex preferences, every Pareto optimum is a Walrasian equilibrium after redistribution of income. Anderson **[8]** shows that, even with nonconvex preferences, with probability one, every Pareto optimum is near a Walrasian equilibrium after redistribution of income. No tractable standard proof is known, even for a special case. As in Perkins **[40]**, the key to the proofs of **[6, 8]** is the fact that one can compare the internal measures of internal sets of Loeb measure zero, and hence it is possible to show in some situations that the measure of an uncountable union of null sets is a null set.

We now turn to a discussion of the book itself. The content and style suggest that the authors' goal was to write the nonstandard version of Royden's *Real analysis* **[48]**. The emphasis is on giving a comprehensive treatment of the mathematical tools used in the nonstandard approach to real analysis, which are common to the applications in probability theory and other areas. The book evolved from graduate courses given by Loeb at the University of Illinois. The treatment is accessible to students with an undergraduate background in real analysis; indeed, we suspect that it could be read by undergraduates who were ready for a Royden-level course. There are many good problems throughout.

The prerequisites from mathematical logic are developed in the first two chapters. Nonstandard models are produced via the ultraproduct construction. The most important idea in NSA is the transfer principle, which asserts that sentences are true in the standard world if and only if they are true, *properly interpreted*, in the corresponding nonstandard world. Specifying the proper interpretation requires an understanding of the distinction between internal objects (those higher-order objects that lie in the ultraproduct) and external objects (subsets of internal objects). Mastering this distinction is the chief hurdle faced by the novice learning NSA. By finessing it, Nelson makes IST easier to learn than NSA, but precludes the development of Loeb measure.

Chapter I begins with a construction of the nonstandard real numbers as an ultraproduct. A very restricted class of first-order sentences, called simple sentences, is then defined. Hurd and Loeb make the transfer principle accessible by first stating and proving it only for simple sentences, where few difficulties arise. They then devote the rest of Chapter I to developing basic properties of the reals (limits, continuity, compactness, and the like) in terms of simple sentences. The advantage of this organization is that it defers most of the technical problems until the reader has had the opportunity to develop intuition, and to realize what the machinery is good for. This approach was inspired by Keisler's nonstandard calculus text and instructors' manual [**26**, **27**].

Fortified by Chapter I, the reader is equipped to plunge into the full development of the transfer principle and other notions such as saturation in Chapter II. This is difficult material for the nonlogician, but less difficult than most mathematicians probably believe. The treatment given here is quite lucid and, at 39 pages, surprisingly short. In technical terms, the models used (denumerably comprehensive enlargements) are not as nice as the κ-saturated enlargements used in Stroyan and Bayod [**52**], whose construction requires an extra step; we think the extra step is worth taking, even if it lengthens the exposition somewhat.

Chapter III presents the nonstandard theory of topological and metric spaces. One of the nicest aspects of the nonstandard theory is its treatment of compactifications. Nonstandard models are in a certain sense universal compactifications with respect to all properties. All the usual standard compactifications (and some new ones—Loeb [**37**]) can be obtained by factoring in an appropriate way.

Chapter IV is the heart of the book, presenting a comprehensive treatment of the nonstandard measure theory, based on the Loeb measure. Until now, this material has only been available in research articles. The authors follow a Daniel-style approach, rather than the Carathéodory-style approach in Loeb's original paper [**36**]. Most mathematicians seem passionately committed to whichever approach to measure theory they first used (or first understood), so some will see this as an improvement, while others would prefer the Carathéodory approach. The last fourteen pages of the chapter give the first hints of the applications to stochastic processes.

For anyone wanting to learn nonstandard analysis, particularly for applications in probability theory, this is an excellent book. A very good one-semester graduate course could be based on this book, supplemented by applications of the instructor's choice.

The other two recent books on nonstandard analysis are also of interest. Stroyan and Bayod's *Foundations of infinitesimal stochastic analysis* [**52**] is a thorough compilation of nonstandard measure-theoretic results selected with an eye to applications in probability theory. It thus covers a great deal of material that is beyond the scope of the Hurd and Loeb book. It contains a quick introduction to nonstandard analysis, but does not try to ease the novice into the subject in the same way as Hurd and Loeb do. It also does not try to describe the applications in probability theory, concentrating (as its title suggests) on underlying machinery. It is probably of greatest value as a reference book, because of its thorough coverage of that machinery. Albeverio, Fenstad, Hoegh-Krohn, and Lindstrom's *Nonstandard methods in stochastic analysis and mathematical physics*, which has not yet appeared, is the only one of these three books that gives an extensive treatment of applications to probability theory and mathematical physics. It provides the best indication of what nonstandard analysis is capable of doing in these areas.

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ROBERT M. ANDERSON