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# STABLE HARMONIC 2-SPHERES IN SYMMETRIC SPACES

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A map $\phi: (M,g) \to (N,h)$ of Riemannian manifolds is *harmonic* if it extremizes the *energy* $E: C^\infty(M,N) \to R$ given (for compact $M$) by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 \text{vol}.$$

A harmonic map $\phi$ is said to be *stable* if the second variation of $E$ at $\phi$ is positive semidefinite. That is: for all smooth variations $\phi_t \in C^\infty(M,N)$ with $\phi_0 = \phi$ we have

$$d^2/dt^2 E(\phi_t)|_{t=0} \geq 0.$$

Of particular interest is the case where $M$ is the sphere $S^2$ and $N$ is a Riemannian symmetric space $G/K$. In this setting harmonic maps are branched minimal immersions, or the finite action solutions of the Euclidean nonlinear $\sigma$-model studied by physicists (see e.g. [**20**] and references cited therein). In the case $G/K$ is Hermitian symmetric it follows from an argument of Lichnerowicz [**9**] that any holomorphic map is energy minimizing in its homotopy class and hence stable. The same is true of antiholomorphic (or $-$holomorphic) maps. The $\pm$holomorphic maps are the *instantons* of the nonlinear $\sigma$-model, and it is important to know if these are the only stable solutions. This is clearly not the case, as one sees by taking $G/K$ to be a product of Hermitian symmetric spaces and by taking a map which is holomorphic into one factor and $-$holomorphic into the other. However, this is the only way a stable map can fail to be $\pm$ holomorphic, as the following theorem shows.

THEOREM 1. *Let $\phi: S^2 \to G/K$ be a stable harmonic map into an irreducible Hermitian symmetric space. Then $\phi$ is $\pm$holomorphic.*

This generalizes a result of Siu and Yau [**16**], who obtained Theorem 1 for the complex projective spaces as targets.

If the target $G/K$ is a general symmetric space $\phi$ can always be lifted to a map into the simply connected covering space. A simply connected symmetric space then splits as a product of irreducible spaces with $\phi$ given by a harmonic map into each factor. As noncompact factors have nonpositive curvature the component of $\phi$ going into such a factor must be constant (by the results of Eells and Sampson [**2**], or more simply by the maximum principle [**4**]), which reduces us to the consideration of compact irreducible symmetric spaces. Moreover $\phi$ is stable if and only if all its components are. We can show that stable harmonic maps into irreducible compact Riemannian symmetric

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spaces always factorize through certain immersed totally geodesic irreducible Hermitian symmetric subspaces. This gives

THEOREM 2. *Let $\phi: S^2 \to G/K$ be a nonconstant stable harmonic map into any Riemannian symmetric space. Then there is a Hermitian symmetric space $G_1/K_1$ totally geodesically immersed in $G/K$ such that $\phi$ factorizes through $G_1/K_1$ as a holomorphic map. Moreover any holomorphic map of a Riemann surface into $G_1/K_1$ gives a stable harmonic map into $G/K$.*

The spaces $G_1/K_1$ are compatible with the decomposition of $G/K$ into irreducible factors, and are irreducible when $G/K$ is. Although there are different submanifolds and complex structures $G_1/K_1$ for different maps, modulo the action of the isometry group $G$, there are only a finite number of possibilities for $G_1/K_1$ and these can be read off the extended Dynkin diagram of $G$. In fact when $G/K$ is irreducible and not itself Hermitian symmetric, the subspaces $G_1/K_1$ all turn out to be complex projective spaces.

Compact irreducible semisimple symmetric spaces can be broken into three classes according to their second homotopy groups $\pi_2(G/K)$: we know the Hermitian symmetric spaces are characterized completely by $\pi_2(G/K) = Z$. The remaining two possibilities are $\pi_2(G/K) = 0$ (this includes all the Lie groups viewed as symmetric spaces) or $\pi_2(G/K) = Z_2$. In the first case there are never any of the Hermitian symmetric subspaces of the $G_1/K_1$ type available and so we have

THEOREM 3. *Let $\phi: S^2 \to G/K$ be a stable harmonic map into a symmetric space with $\pi_2(G/K) = 0$. Then $\phi$ is constant.*

In the remaining case where $\pi_2(G/K) = Z_2$, when $G/K$ is irreducible the various possible inclusions $G_1/K_1 \subset G/K$ induce surjections on $\pi_2$, so we obtain

THEOREM 4. *If $G/K$ is a symmetric space then every homotopy class of maps $S^2 \to G/K$ has a stable harmonic representative.*

REMARKS. 1. The irreducible symmetric spaces with $\pi_2(G/K) = Z_2$ are $\mathrm{SU}(n)/\mathrm{SO}(n)$, $n \geq 3$, $\mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$, $p \geq q \geq 3$, together with the following exceptional spaces, which we list along with the largest of the $G_1/K_1$ subspaces for each:

$$E_6/\mathrm{Sp}(4)^{\sim},\ \mathrm{CP}^2;\quad E_6/\mathrm{SU}(6)\mathrm{Sp}(1),\ \mathrm{CP}^4;\quad E_7/\mathrm{SU}(8)^{\sim},\ \mathrm{CP}^5;$$
$$E_7/\mathrm{Spin}(12)\mathrm{Sp}(1),\ \mathrm{CP}^6;\quad E_8/\mathrm{Spin}(16)^{\sim},\ \mathrm{CP}^8;\quad E_8/E_7\mathrm{Sp}(1),\ \mathrm{CP}^7;$$
$$F_4/\mathrm{Sp}(3)\mathrm{Sp}(1),\ \mathrm{CP}^2;\quad G_2/\mathrm{SO}(4),\ \mathrm{CP}^2.$$

Here $K^{\sim}$ denotes a quotient $K/Z_2$. The spaces in this list where $K$ is not simple are the quaternionic Kähler manifolds studied by Wolf [**19**].

2. Sacks and Uhlenbeck [**14**] show by general methods that $\pi_2(G/K)$ has a set of stable harmonic generators; we can exhibit explicit generators which are embedded homogeneous 2-spheres arising from suitably chosen root spaces. When $\pi_2(G/K) = Z_2$ we can precompose such stable harmonic maps with holomorphic maps of $S^2$ of arbitrarily high degree without losing stability and

so represent the trivial and nontrivial homotopy classes by nonconstant stable harmonic maps of arbitrarily high energy.

3. The result of Theorem 3 on the nonexistence of nonconstant stable harmonic maps follows by case-by-case checking from the work of Howard and Wei [**6**], Ohnita [**12**], Tyrin [**18**] (for the classical spaces), and Smith [**17**], since the list of irreducible spaces (of type III) with $\pi_2(G/K) = 0$ coincides with the list

$$S^n, \quad \mathrm{SU}(2n)/\mathrm{Sp}(n), \quad \mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q), \quad E_6/F_4, \quad F_4/\mathrm{Spin}(9)$$

of spaces with unstable identity maps. However the Lie groups with stable identity maps still have no nonconstant harmonic 2-spheres as a consequence of Theorem 3. This is also a consequence of a construction by Uhlenbeck of a canonical energy-decreasing variation of a nonconstant harmonic map. See also Pluzhnikov [**13**] for related results on the instability of the identity map.

4. Micallef [**10**] has recently shown the existence of nonconstant harmonic maps $S^2 \to G$ of index $\leq 1$. As a consequence of Theorem 3 these maps are seen to have index precisely equal to 1.

5. For maps of $S^2$ the index of the energy agrees with the index of the area, so all the above theorems apply with harmonic maps replaced by minimal 2-spheres.

6. Our method depends in an essential way on the structure theorem of Grothendieck [**5**] concerning holomorphic bundles over the Riemann sphere. For Riemann surfaces of higher genus we can obtain the results of Siu [**15**] and Leung [**8**] for complex projective spaces or $n$-spheres respectively, with the same restrictions on the branching of the map. By using a twistor space over the domain manifold, Burns and De Bartolomeis have recently shown that all harmonic maps of a Riemann surface of any genus into a complex projective space must be $\pm$holomorphic.

7. The index of unstable maps is not known in general, but Ejiri [**3**] has shown that any full harmonic map from $S^2$ to $S^{2n}$ has index at least $2n(n+2)-6$.

SKETCH OF THE PROOFS. We identify the tangent bundle to $G/K$ with the orthogonal $[\mathfrak{p}]$ of the bundle $[\mathfrak{k}]$ of Lie algebras of the stability subgroups in the trivial bundle $\mathfrak{g}$. The Levi-Civita connection in $[\mathfrak{p}]$ induces a connection in $[\mathfrak{k}]$ and so the direct sum connection in $\mathfrak{g}$. This may be pulled back to $S^2$ by a map $\phi\colon S^2 \to G/K$ to give a connection $\nabla^\phi$ in the trivial bundle $\mathfrak{g}$ of Lie algebras over $S^2$. The theorem of Koszul and Malgrange [**7**] gives the complexification $\mathfrak{g}^c$ the structure of a holomorphic bundle of Lie algebras with the $(0,1)$ part of $\nabla^\phi$ being the $\overline{\partial}$-operator. Grothendieck's structure theory [**5**] tells us this bundle contains a holomorphic bundle of parabolic subalgebras $\mathfrak{q}$ compatible with the decomposition $\mathfrak{g}^c = \phi^{-1}[\mathfrak{p}]^c + \phi^{-1}[\mathfrak{k}]^c$. This bundle of parabolics can be modified as in [**1**] so that its nilradical bundle $\mathfrak{n}$ is generated by its intersection with $\phi^{-1}[\mathfrak{p}]^c$ and this intersection is contained in the subbundle of $\phi^{-1}[\mathfrak{p}]^c$ generated by holomorphic sections. On the other hand Moore [**11**] gives the Hessian of the energy as

$$\mathrm{Hess}(u,v) = 4\int_{S^2} (\nabla^\phi_{\bar z} u, \nabla^\phi_{z} v) - ([\delta, u], [\overline{\delta}, v]) i/2\, dz \wedge d\bar z,$$

where $u$ and $v$ are sections of $\phi^{-1}[\mathfrak{p}]^c$ and $\delta$ is $\phi * \partial/\partial z$ viewed as a section of $\mathfrak{G}^c$. Thus if $\phi$ is stable, taking $\nabla^{\phi}_{\bar{z}} u = 0$ we see that $\delta$ must commute with all holomorphic sections of $\phi^{-1}[\mathfrak{p}]^c$ and hence with the nilradical bundle $\mathfrak{n}$. But $\delta$ takes its values in $\mathfrak{n}$, and hence must be in the center $\mathfrak{z}$ of $\mathfrak{n}$. A Lie-theoretic argument shows that when $G$ is simple any parabolic has an irreducible action on the center of its nilradical. If $G/K$ is Hermitian symmetric the parabolic is compatible with the complex structure and so $\mathfrak{z}$ must be contained in either the $(1,0)$ or $(0,1)$ vectors giving Theorem 1. Theorem 2 follows by observing that $\mathfrak{z}+\bar{\mathfrak{z}}$ gives rise to a Lie triple system such that $\mathfrak{z}$ is tangent to an immersed totally geodesic Hermitian symmetric subspace $G_1/K_1$ through which $\phi$ can be shown to factorize holomorphically as a consequence of the vanishing of certain holomorphic differentials. The Dynkin diagram of $G_1$ is obtained from the extended Dynkin diagram of $G$ by striking out the complementary simple roots of $\mathfrak{q}$ and taking the connected component of the negative of the highest root in what remains. Theorem 3 is obtained by showing that when $\pi_2(G/K) = 0$, $\mathfrak{z}$ is always in the stability subalgebra. This forces $\delta$ to vanish and so $\phi$ is constant. Theorem 4 also follows since we can describe the $G_1/K_1$ subspaces so explicitly.

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