boundaries do not satisfy the relevant curvature conditions. Solutions with infinite boundary data are also considered. This sometimes produces interesting generalizations of one of Scherk's classical minimal surfaces.

The final chapter of this book is devoted to extensions of the theorem of S. Bernstein that a function $z = f(x, y)$ satisfying the minimal surface equation and defined for all (x, y) in $\mathbf{R}^2$ must be affine. The corresponding theorem for functions $f\colon \mathbf{R}^n \to \mathbf{R}$ is true when $n = 3, 4, 5, 6, 7$ and fails for larger n.

As indicated above, this book leads one near the frontiers of knowledge in the study of oriented area-minimizing hypersurfaces. Much more remains to be done. For example, we know very little about the structure of singularities—not even if they necessarily have integer dimensions or whether or not they can persist under small boundary deformations.

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Distribution of values of holomorphic mappings, by B. V. Shabat, translated from Russian by James R. King. Translations of Mathematical Monographs, vol. 61, American Mathematical Society, Providence, R. I., 1985, v + 225 pp., $79.00. ISBN 0-8218-4514-4

Value distribution theory has known alternating periods of quiescence and rapid progress: the classical function-theoretic work of Nevanlinna, Ahlfors' introduction of differential-geometric methods, the work of Stoll, and the work of the Griffiths school, motivated by problems in algebraic geometry.

Shabat's book, translated from the Russian by James King, is an excellent self-contained exposition of this last phase. It contains an extensive bibliography, but, sadly, no index.

Function theory. According to the fundamental theorem of algebra, a polynomial equation $f(z) = a$ of degree d has precisely d solutions (counting multiplicities) for all a. Classical value distribution theory seeks similar regularity properties of the solution sets $f^{-1}(a)$ when f is holomorphic, or more generally, meromorphic. The transcendental equation $e^z = a$ shows that (i) there may be lacunary values (e.g. $a = 0$) for which there are no solutions and that (ii) the solution set is in general infinite. One therefore speaks of the order of growth of the solution set, measured by the "counting function" $n_f(r, a) = \#\{z \mid f(z) = a,\ |z| < r\}$, and the order of growth of a function, measured by an averaged modulus indicator,

$$t_f(r) = \int_{|z|=r} \log^+ |f(z)|\, d\theta.$$

Somewhat better behaved are the logarithmic averages $N_f(r, a)$, $T_f(r)$, where in general the logarithmic average of a function $g(t)$ is the integral

$$G(t) = \int_0^t [g(s) - g(0)]\, ds/s.$$

The fundamental regularity results, the first and second main theorems of classical Nevanlinna theory (FMT and SMT) assert that

$$N_f(r, a) \leqslant T_f(r) + O(1),$$

$$(q-2)T_f(r) + N_f(S, r) \leqslant \sum_{j=1}^{q} N_f(a_j, r) + O(\ln T_f(r)),$$

where $n_f(S, r)$ is the counting function for the ramification divisor, and $N_f(S, r)$ is its logarithmic average. By ramification divisor we mean the analytic set defined by the vanishing of the derivative of f, with points where f' vanishes to order n being counted n times. The FMT generalizes the inequality (number of zeros) $\leqslant$ degree, a fact which is perhaps made more evident by Croftons' formula:

$$T_f(r) = \int_{\mathbf{P}^1} N_f(r, a)\, da,$$

where $\mathbf{P}^1$ is complex projective space of dimension one (the Riemann sphere), and where da is the invariant measure on $\mathbf{P}^1$ of unit total mass. The order function T_f is thus revealed as a kind of integrated degree.

To obtain more quantitative results define the *f-defect* of a to be

$$\delta_f(a) = 1 - \overline{\lim_{r \to \infty}} \frac{N_f(a, r)}{T_f(r)}$$

so that (by the FMT) $0 \leqslant \delta_f(a) \leqslant 1$, while $\delta_f(a) = 1$ for lacunary values. The SMT—the more subtle of the two results—then gives the celebrated defect inequality

$$\sum_a \delta_f(a) \leqslant 2,$$

a result which one should view as a quantitative version of Picard's theorem: the number of lacunary values (including infinity) cannot exceed 2, with the exponential function achieving the bound.

Differential geometry. Although the function-theoretic proofs gave the best possible result for entire meromorphic functions, they did not, unfortunately, provide a satisfactory understanding of the constant 2 in the right-hand side of the defect inequality. An explanation of this maximum defect came in 1937 with Ahlfors' introduction of differential-geometric methods to the subject, an introduction which was to play a crucial role in all future developments [**A1**, **A2**].

Ahlfors begins with a more geometric definition of the order function. Let

$$\omega = \frac{i}{2\pi} \frac{i\, dz \wedge \overline{dz}}{\left(1 + |z|^2\right)^2},$$

be the volume form on the Riemann sphere associated to the Fubini-Study metric: the unique invariant volume form of total mass 1. Define a new order function, one which measures the area of the image of the disk of radius r, by

$$t_f(r) = \int_{\|z\| \leqslant r} f^*\omega.$$

The logarithmic average of this function differs from the one previously defined by a bounded function.

To establish the FMT in this context, consider the function

$$v_a = \frac{1 + |z|^2}{|z - a|^2}.$$

An easy calculation shows that

$$(i/2\pi)\partial\bar{\partial} \log v_a = \omega - \delta_a,$$

where δ_a is the Dirac current with support at a. By this we mean the differential 2-form with distribution coefficients whose integral against a test function ρ has the value $\rho(a)$. Take the pullback of the preceding relation along a meromorphic function f, integrate over the disk of radius r, apply Stokes' theorem, and then form the logarithmic average. The result is the FMT.

A preliminary form of the SMT is obtained, very roughly, by applying the process just outlined to the Fubini-Study metric, modified to have singularities of the right kind at the points $a_1, \ldots, a_q$. In more detail, set

$$ds^2 = \left[\prod_{i=1}^{q} v_a^\lambda\right]\left[\frac{dz \otimes \overline{dz}}{(1+|z|^2)^2}\right] = h\, dz \otimes \overline{dz},$$

where $0 < \lambda < 1$. Write $f^* ds^2 = h_0\, dz \otimes \overline{dz}$ and compute the Laplacian to get

$$\frac{i}{2\pi}\partial\bar{\partial}\log h_0 = -f^*\Omega + \lambda q f^*\omega + \delta_S - \lambda \sum_{i=1}^{q} \delta_{f^{-1}(a_i)},$$

where Ω is the *curvature form* of the Fubini-Study metric, and where the δ's refer to the Dirac currents of the indicated sets, with S as the ramification locus. Because $\Omega = K\omega$, where K is the Gaussian curvature, equal to 2 in this case, the integral of the above equation over the disk of radius r gives

$$\frac{i}{2\pi}\int_{\Delta_r} \partial\bar{\partial}\log h_0 = (\lambda q - 2)\int_{\Delta_r} f^*\omega + n_f(S, r) - \lambda \sum_{i=1}^{q} n_f(a_i, r).$$

Apply Stokes' theorem, form the logarithmic averages, and work to show that the resulting left-hand side is bounded above by a function which is of the order of $\log T$. In the limit of $\lambda = 1$ this gives the SMT.

The preceding argument reveals the origin and significance of the maximum defect: it is geometric, given concretely by the Gaussian curvature K. The curvature can be further interpreted as the topological Euler characteristic of the Riemann sphere (i.e. the integrated curvature) or, usefully but perhaps less transparently, as the proportionality constant between the curvature of the tangent bundle and the curvature of the hyperplane bundle.

Closely related to the above was Ahlfors' differential-geometric proof of the Schwarz lemma, a proof which in the late 1960s and early 1970s played a central role in complex algebraic geometry (Griffiths' classifying spaces for Hodge structures and properties of the period mapping).

Algebraic geometry. Chow's theorem asserts that any closed analytic subvariety of complex projective space is algebraic. Consider therefore an analytic subvariety Z of dimension q in $\mathbf{C}^n = \mathbf{P}^n - \{\text{hyperplane}\}$. Bishop **[B]** and Stoll **[S2]** showed that if ω is the standard Kähler form, given by $\omega = i\partial\bar{\partial}\log\|z\|^2$, and if $\int_Z \omega^q < \infty$, then the closure of Z is an analytic, and hence algebraic, subvariety of projective space. Furthermore, arguments based on the Chern character, which connects cohomology and K-theory, and Grauert's theorem, which asserts that a complex vector bundle can be deformed to a holomorphic one, show that any homology cycle of even dimension in an affine algebraic variety can be represented by an analytic cycle: a formal linear combination of analytic subvarieties. Now an affine variety (which is a Stein space) is just the complement of a hyperplane in an algebraic variety. Thus, if the analytic cycle has finite volume, its closure will be algebraic.

This line of reasoning led Griffiths to formulate a plan for proving the Hodge conjecture: Given a primitive integral cohomology class of type (p, p), represent its restriction to a hyperplane complement by an analytic cycle, and then develop an obstruction theory relating Hodge type to order of growth of the volume of the cycle, so that (p, p) classes can ultimately be represented by analytic cycles of finite volume **[CoG]**.

Although this plan is still unrealized, it, with Ahlfors' work, prepared the soil for the modern flowering of value distribution theory. Among the first results were generalizations of Picard's theorem and then the defect relations to the context of equidimensional mappings of several complex variables **[CaG]**, i.e., holomorphic mappings $f\colon \mathbf{C}^n \to \mathbf{P}^n$. Define the counting function of an analytic set of dimension q,

$$n(Z, r) = \int_{Z[r]} \omega^q,$$

where $Z[r] = \{a \in Z \mid \|a\| \leqslant r\}$. Thus, if A is a smooth algebraic hypersurface of degree d, then one may consider $n(f^{-1}(A), r)$ and seek for it first and second main theorems. The consequent defect relations assert that if the Jacobian determinant of f is not identically zero, then

$$\sum_A \delta(A) \leqslant \frac{n+1}{d},$$

where the hypersurfaces A meet with normal crossings, i.e. locally as do the coordinate hyperplanes in $\mathbf{C}^n$. In particular, if f has $n + 2$ lacunary hyperplanes then its Jacobian determinant must vanish identically, a generalization of Picard's theorem to several complex variables (in this context a three-point set is a smooth hypersurface of degree three in $\mathbf{P}^1$). For the proof one follows Ahlfors' lead and constructs a volume form on projective space which (a) has prescribed (negative) curvature properties and which (b) has singularities of the correct type along the hypersurfaces A. As a model of both the curvature and singularity properties one takes the Poincaré volume form on a product of punctured disks, where that given on a single punctured disk is

$$\omega = \frac{i\,dz \wedge \overline{dz}}{\|z\|^2 (\log\|z\|^2)^2}.$$

The number on the right-hand side, as in all results of this type, is a proportionality constant relating the curvature forms of the various line bundles used in the construction of the volume forms.

Higher codimension. With the study of analytic sets of codimension greater than one came results with no classical analogue. In algebraic geometry Bezout's theorem asserts that if A_1 and A_2 are projective algebraic curves with no components in common, then

$$\deg(A_1 \cap A_2) = \deg(A_1)\deg(A_2),$$

where on the left the degree is the number of points counting multiplicities and where on the right the degree is the degree of the defining equation. In analytic geometry one expects an analogous inequality, with the integrated counting function in the place of the degree. In the simplest instance the sets A_i are defined by a single analytic function, and so the intersection is defined by a vector-valued function. Thus, the hoped-for inequality would follow from a suitable vectorial version of the FMT. Cornalba and Shiffman, however, found functions $f\colon \mathbf{C}^2 \to \mathbf{C}^2$ which show this to be impossible: f has moderate growth while the analytic set $f^{-1}(0)$ has arbitrarily high growth rate. The problem was that while there was a FMT, it contained a term which could not be estimated in terms of the order function T. Subsequent work showed that $f^{-1}(a)$ is well-behaved for all a outside a thin set (a set of capacity zero). The "bad term" of the FMT, while inevitably present, was well-behaved in the mean.

This essentially different (and more difficult) behavior of analytic sets of codimension greater than one is mirrored by similar phenomena in algebraic geometry, where (for example) one knows the Hodge conjecture in codimension one, but not in higher codimension.

Conclusions. A single paradigm dominated the work of this last period: find a (tensorial) potential S with suitable singularities and curvature properties, compute $\partial\bar{\partial}S$ to obtain a C^∞ term (the curvature) and a singular term (the integration current of the analytic set to be "counted"), apply Stokes' theorem, integrate, form a logarithmic average, and (attempt to) bound any terms other than the order and counting functions by $O(\log T)$. If successful, obtain a good FMT (and perhaps SMT). The influence of Ahlfors' proof of Nevanlinna's theorem is clear.

New techniques or new problems may once again reinvigorate the subject. The recent connection with number theory is perhaps a sign of this [**V**, **L**].

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