technique to the Dirac operator, Lichnerowicz showed that if a compact spin manifold M^{4k} of dimension $4k$ has positive scalar curvature, its only harmonic spinor is zero, from which it follows by the Atiyah-Singer index theorem that its $\hat{A}$-genus is zero. Thus positive scalar curvature could have a topological implication. Following a question of Kazdan-Warner, R. Schoen and S. T. Yau, using minimal surfaces, proved that the three-dimensional torus T^3 (as well as many other three-manifolds) does not have a Riemannian metric of positive scalar curvature. M. Gromov and B. Lawson showed that this is also true for the n-dimensional torus T^n, by using the Dirac operator.

Generalizing the two-dimensional case, Yamabe had the idea of attaining a constant scalar curvature metric on a compact manifold M^n of dimension $n \geqslant 3$ by a pointwise conformal transformation. Important progress on this problem, the Yamabe problem, was achieved by T. Aubin and the work was recently completed by R. Schoen. The result is a positive answer to the Yamabe problem.

Concerning the Ricci curvature, the most important result is perhaps the Calabi conjecture, which was proved by S. T. Yau. Given a compact Kähler manifold M, its Ricci form is of type $(1,1)$ and belongs to the first Chern class $c_1(M)$ in the sense of de Rham cohomology. The Calabi-Yau theorem says that any closed form of $c_1(M)$ can be realized as the Ricci form of a Kähler metric of M. The theorem has many consequences. The proof of the theorem can now be simplified, using an interior estimate of L. C. Evans.

Another nice result on the Ricci curvature is the following theorem of R. Hamilton: Let (M^3, g_0) be a compact 3-manifold with positive Ricci curvature. Then there is a family of metrics g_t, $0 \leqslant t \leqslant \infty$, with positive Ricci curvature, where g_∞ is an Einstein metric and hence has constant positive sectional curvature.

The above are some highlights of the monograph. The monograph is informative, up-to-date, and contains a list of open problems. A natural question is: why Riemannian? A study of analogous problems on Finslerian manifolds, which are based on more general variational problems, will not only open to a new horizon, but also give a better understanding of the Riemannian case itself.

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Infinite dimensional Lie algebras, by Victor G. Kac, second edition, Cambridge University Press, 1985, xvii + 280 pp., $24.95. ISBN 0-521-32133-6

The first edition of this book was published by Birkhäuser-Boston in 1983. It is testimony to the broad interest of the subject, barely twenty years old, and to the merits of the book that 1985 already saw the appearance of a second edition. There are no major changes or additions here from the first edition, only corrections and updates of the bibliography. Reviews of the first edition appeared at least in the Zentralblatt V. 537, review #17001 (B. Weisfeiler) and in the Bulletin of the London Math. Society **17** (1985), 401–404 (R. Carter). The former is rather brief. The review by Carter stresses connections with other branches of mathematics, and that is without doubt the feature of the subject that gives the book its importance. Such connections are dense throughout roughly the last hundred pages of text.

The beginnings of the subject were not thus motivated. When Serre gave his presentation for finite-dimensional complex semisimple Lie algebras (see [**14** or **4**]), R. V. Moody was a graduate student at Toronto. It appears to have struck him and/or M. Wonenburger, his adviser, as an interesting question to ask what happens if the conditions on the Cartan matrix (a_{ij}), that determines the Serre relations, are weakened. Moody was able to write a fine thesis on this question [**11**, **12**]. Particularly interesting was the case where, in addition to rather natural assumptions, the matrix (a_{ij}) is singular but each proper principal submatrix is one of those considered by Serre. Moody [**13**] and other early authors referred to this as the "Euclidean" case, but now "affine" seems the preferred term.

At the same time, Lie algebras with a **Z**-grading were under study in Moscow. This work seems mainly to have been motivated by efforts to characterize, or narrowly to enlarge to an axiomatically defined class, the graded Lie algebras associated with simple Cartan pseudogroups (see, e.g. [**15**]). Apparently as an afterthought to a paper aimed at such a characterization, Kantor observed that, in a "generic" case, a subset of the Serre relations already presented a simple graded Lie algebra [**6**]. Meanwhile, Kac was studying simple, or nearly simple **Z**-graded Lie algebras $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ such that $d(n) = \sum_{|i| \leqslant n} \dim \mathfrak{g}_i$ grows polynomially with n. There were additional technical conditions, among them that $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ generates $\mathfrak{g}$ and that the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is faithful and irreducible. (In the case of *linear* growth of $d(n)$, O. Mathieu has recently shown these restrictions to be unnecessary.) An important device used by Kac was that if elements $e_1, \ldots, e_r$ span $\mathfrak{g}_1$; $f_1, \ldots, f_r$ span $\mathfrak{g}_{-1}$; and if $\mathfrak{g}_0$ contains commuting elements $h_1, \ldots, h_r$ with $[e_i f_j] = \delta_{ij} h_i$, $[h_i e_j] = a_{ji} e_j$, $[h_i f_j] = -a_{ji} f_j$, then for "most" matrices (a_{ij}) the universal graded algebra with $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ as its part in these dimensions is of faster-than-polynomial growth and has no proper graded ideal intersecting $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ trivially. (Such a universal algebra was also constructed by Kantor, and it is the algebra he proved simple.)

This observation made it possible for Kac to rule out many configurations, and ultimately to solve his problem. The solution had to include more than Cartan pseudogroups and finite-dimensional Lie algebras; Kac noted that if t is an indeterminate and $\mathring{\mathfrak{g}}$ is a finite-dimensional complex simple Lie algebra, then $\mathbf{C}[t, t^{-1}] \otimes_{\mathbf{C}} \mathring{\mathfrak{g}}$, graded by powers of t, satisfies the condition (with $d(n)$ linear), as do some "twists" of this construction, associated with a symmetry of the Dynkin diagram of $\mathring{\mathfrak{g}}$. These algebras have generating sets $\{e_i, f_i, h_i\}$ as above, where the matrix (a_{ij}) is one of the "affine" ones, classified independently by Kac and by Moody. The additional Serre relations hold, but do not present Kac's simple algebras; Moody's algebras here contain a nontrivial central combination of the h_i. (Final clarification of the relation between the constructions was only given by Gabber and Kac in 1981 [3].)

In 1970, the new infinite-dimensional Lie algebras did not seem like much more than curiosities. They had root systems and Weyl groups, and these "affine" systems had also turned up in the work of Iwahori and Matsumoto on p-adic Chevalley groups [**5**]. They were important enough to be incorporated, as abstract root systems, into the treatise of Bourbaki [**1**]. Then, in 1972, I. G. Macdonald observed that if one pretended that an affine root system was the root system of a Lie algebra whose trivial module satisfied a seemingly natural analogue of Weyl's character formula, specialization of power-series variables yielded identities involving Dedekind's eta-function.

It was not quite so neat as that. The known identities were only duplicated if one modified the "product side" by a factor that turned out to be an infinite product of polynomials in $e^{-\delta}$, where δ is a minimal positive "*null*" root ($\langle \delta, h_i \rangle = 0$, all i). Macdonald was able to give a combinatorial proof of his corrected identity, based on the finite-dimensional theory.

The intervening time had not been one of inactivity—for example, Moody and Teo had studied the adjoint Chevalley group corresponding to one of the new Lie algebras. But it is probably correct to say that the results of Macdonald gave the impetus to the swell of activity that we are still experiencing. Kac, Moody, and Garland, each in his own way, set about explaining Macdonald's identities via characters of representations. All were led to consider highest-weight modules, but without the "Harish-Chandra homomorphism" at their disposal. Kac used Bernstein-Gelfand-Gelfand resolutions, Moody used a suitable Casimir operator, and Garland showed that one could develop a complete analogue of Bott's theory of $\mathfrak{n}^+$-cohomology for the trivial module. This was then extended by Garland and Lepowsky to other highest-weight modules. The result in each case was a "natural" setting for Macdonald's formulas, with a natural explanation for the mysterious factor. Highest-weight modules were also being studied by Marcuson, as natural modules for the action of groups like Chevalley groups.

A number of the ideas and insights that went into this work offer something new even in the case of finite-dimensional semisimple Lie algebras. In fact, the twisted ("$k > 1$", in the author's notation) algebras supplied Kac with an effective tool for classifying automorphisms of finite period in the classical case.

A rough approximation to the first ten chapters of the book (of fourteen in all) could be given by saying that they present the ideas and results of the theory up to the historical point now reached in this account. To leave it at that would be unfair; the chapters are improved by all the insights, extensions, and embellishments that the author and others have added. The main points are developed quite carefully and with consideration for a reader who may be a graduate student. Changes from the first edition include occasional sentences to clarify terminology, correction of most typographical errors, and a general addition of touches to make the work more understandable. In his introduction, the author notes that "The book was prepared using D. Knuth's TEX."

Side-by-side with the first edition, the second edition looked rather smudgy at first. With time, this annoyance passed.

Each chapter closes with a set of exercises and a set of bibliographic notes. Often the exercises lead the reader to results in papers cited in the bibliography, and the sources of the exercises are identified. The diligent reader can thus attain a very solid foundation in the general theory as it stands today.

There is one change to be noted in particular. An extra hypothesis must be added to obtain the conclusions of Lemma 9.10 b) of the first edition, and this has been done. An exericse (9.15) shows that the added hypothesis is needed.

The simplest new algebra, (an extension of) that above where $\mathring{\mathfrak{g}} = sl(2)$, already yields, for modules of highest weight zero, the Jacobi triple product identity (Exercise 10.9). Exciting prospects regarding connections with partition-identities, modular forms, etc., immediately suggested themselves as applications of the character formula to various algebras and modules. The work of Lepowsky, of Feingold and Lepows and of Lepowsky and Milne in this direction prior to 1978 is summarized in [7]. An attractive target for application of the theory was the Rogers-Ramanujan identities, on which Lepowsky and Wilson have written at length [**8**, **9**, **10**]. Kac also appears to have been involved in this work at an early stage (see below). A selection of related topics is treated in Chapters 11–13, many of these topics the results of work of the past ten years by Kac and/or D. Peterson. Among the citations are some references to connections with the Fischer-Griess "monster" of the list of finite simple groups—cf. [**2**].

Chapter 14 is an elegantly presented account of how one is led, via realizations as linear transformations of polynomial algebras in infinitely many generators and the decomposition of the tensor squares of such representations, to matters such as the Korteweg-deVries equation. The work summarized here is mainly that of the Kyoto school, sometimes involving "affine" Lie algebras with an *infinite* Cartan matrix. Soliton solutions to the nonlinear PDE are constructed by combining the Hirota transformation with the action of the associated Chevalley group on a highest-weight vector. (See also [**17** and **18**].)

In the representation on polynomials, as in a number of the applications to combinatorics, an important tool is the "vertex operator", found already in the physical papers of Mandelstam and Schwarz from the mid-seventies on dual resonance models. Roughly, one identifies an infinite Heisenberg subalgebra within the Kac-Moody algebra, and the Heisenberg generators act canonically as creation and annihilation operators on the polynomials. (There are different ways of realizing such a Heisenberg subalgebra, carrying labels such as "principal" and "homogeneous".) The vertex operators then provide a means for extending the representation to the whole Kac-Moody algebra.

The further striking interactions with physics and mathematics are too numerous for me to avoid slighting many works if I try to mention more. Authors who have contributed and are not mentioned will recognize that they are in excellent company. An idea of the level of activity in and related to the field is given by the fact that the author's bibliography lists about 275 entries, of which probably not more than 50 existed ten years ago. About 85 of these are new to the second edition, or updated from the first edition. (There have also been a few deletions.) For more references, there is *A Kac-Moody bibliography and some related references*, compiled by Benkart at Wisconsin, and published as [**16**], with perhaps 350 entries. This bibliography is somewhat more thorough than that of Kac on the side of combinatorics and physics. Some 205 of the entries in the Benkart bibliography, not including preprints, do not appear in the Kac bibliography, so the combined coverage is quite thorough. (It should be noted that Kac seldom lists research announcements when a fuller account has appeared—the Benkart bibliography contains at least five entries with Kac as author or co-author that are not in his bibliography.)

In a burgeoning field of research, where the next challenging hill is begging all to try to climb it, it is inevitable that questions of priority will arise. Perhaps those that have come up in this area have been exacerbated by differences in the handling of priority matters in the mathematical cultures where the leading workers developed. Thus one is sad to see no acknowledgement of the fact, known to the author at that time, that important results in the paper referred to as Kac [**1978** A] were being presented in lectures by Lepowsky in 1977, and to read the author's fluttering efforts in the priority race on Rogers-Ramanujan at the bottom of p. 147 and the top of p. 148. In some instances a more subtle torsion is involved. Regardless of intention, the effect is to slight the work of quite a few fine mathematicians. These circumstances mar this excellent book, and they may have impeded general access to it. They do not change my enthusiastic recommendation, but they urge on me the audacity to end this review with a benediction:

May the spirit of fellowship in our wonderful common enterprise drive out all that is bitter and base, and may you work on in graciousness and generosity.

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Hardy classes and operator theory, by Marvin Rosenblum and James Rovnyak, Oxford Mathematical Monographs, Oxford Univ. Press, New York and Clarendon Press, Oxford, 1985, xii + 161 pp., $39.95. ISBN 0-19-503591-7.

Hardy space theory has its classical origins in the work of G. H. Hardy and the brothers Riesz, but the modern origins of the subject begin with the theorem of A. Beurling in 1949. The Hardy space H^2 is defined to be the space of functions f analytic on the unit disk such that

$$\|f\|_2^2 = \sup\left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2\, dt \colon 0 \leqslant r < 1 \right\} < \infty.$$

The theorem of Beurling asserts that any such f has an inner-outer factorization $f = bg$ where b is an inner function and g is an outer function. By definition an inner function is a function analytic on the unit disk whose nontangential boundary values have modulus 1 almost everywhere on the unit circle. An outer function can be defined as the solution of the extremal problem of finding the function g in H^2 that maximizes $|g(0)|$ among all functions with $|g(e^{it})|$ equal to a prescribed function on the boundary. Both inner functions and outer functions have finer structure; an inner function can be factored further as the product of a Blaschke product and a singular inner function while an outer function is characterized by having an integral representation of a certain form. It was recognized already by Beurling that this purely function-theoretic result has connections with operator theory. Indeed, from this theorem one can classify all the closed invariant subspaces for the