BOOK REVIEWS

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User's guide to spectral sequences, by John McCleary, Mathematics Lecture Series No. 12, Publish or Perish, Inc., Wilmington, Delaware, 1985, xiii + 423 pp., $40.00. ISBN 0-914098-21-7

What are spectral sequences, and what are they good for? Perhaps we can best answer this question by giving a couple of examples.

First example, from homological algebra. Let G be an arbitrary group, and let A be a G-module. To understand this example, the reader needs to know a little bit about the cohomology groups of G with coefficient module A, denoted by $H^q(G, A)$, $q = 0, 1, 2, \ldots$. In the study of these cohomology groups the following problem arises: Let N be a normal subgroup of G; assume we know the cohomology groups of the subgroup N and the quotient group G/N (with some suitable choices of coefficient module). With this information, can we determine the cohomology groups of G? A little thought should convince the reader that, in general, the answer will probably be No, because usually there exist many different, nonisomorphic "extensions" G, given N and G/N. In other words, more information will be needed.

Thus a more reasonable problem is the following: Determine what relations must exist between the cohomology groups of N, G/N, and G. The answer to this problem is given by a spectral sequence. We will now explain what this spectral sequence is.

The reader will recall that the cohomology groups of G are defined by means of a certain "cochain complex," which consists of a sequence of abelian groups $\{C^q \mid q = 0, 1, 2, \ldots\}$ together with a "coboundary operator," δ, which is a homomorphism $\delta: C^q \to C^{q+1}$ defined for all $q \geqslant 0$ and having the basic property that $\delta \circ \delta = 0$. The subgroup of C^q which is the kernel of δ is denoted by Z^q, and the image subgroup $\delta(C^{q-1})$ is denoted by B^q. From the basic property $\delta \circ \delta = 0$, it follows that $B^q \subset Z^q$; and the quotient group, Z^q/B^q, is by definition the qth cohomology group, H^q.

In connection with spectral sequences the following alternative terminology for these concepts has become common: the sequence of abelian groups $\{C^q\}$ is called a "graded abelian group", and the homomorphism δ is said to have "degree $+1$", because $\delta(C^q) \subset C^{q+1}$, and to be a "differential" because $\delta \circ \delta = 0$. The following slight generalization of these concepts has also become standard: a *bigraded* group is a doubly indexed sequence of abelian groups,

$$E = \{E^{p,q}\}$$

where p and q may run through all integers, or all nonnegative integers. A homomorphism $d: E \to E$ is said to have bidegree (r, s) if for each (p, q), d is a homomorphism of $E^{p,q}$ into $E^{p+r,q+s}$. As before, such a homomorphism is said to be a "differential" if it satisfies the condition $d \circ d = 0$. If this is the case, we can define the nested sequence of subgroups $B^{p,q} \subset Z^{p,q} \subset E^{p,q}$, as before, and then define

$$H^{p,q}(E, d) = Z^{p,q}/B^{p,q}.$$

Note that $H(E, d) = \{H^{p,q}(E, d)\}$ is again a bigraded group; it is sometimes called the "derived group."

A *spectral sequence* is an infinite sequence of bigraded (or graded) groups with differentials,

$$(E_1, d_1), (E_2, d_2), (E_3, d_3), \ldots$$

with the property that

$$E_{n+1} = H(E_n, d_n)$$

for $n = 1, 2, 3, \ldots$. In other words, the derived group of each term of the series is the group of the next term. Its differential, d_{n+1} is *not* determined by (E_n, d_n), however.

We can now explain the Lyndon-Serre-Hochschild spectral sequence. As above, let G be a group, N a normal subgroup, and A a G-module. Note that A is also an N-module, by restricting the action of G on A to the subgroup N; hence the cohomology groups $H^q(G, A)$ and $H^q(N, A)$ are defined for all $q \geqslant 0$. Note also that the action of G on N by conjugation gives $H^q(N, A)$ the structure of a G-module; however, it can be shown rather easily that the restriction of this action to the normal subgroup N is trivial, hence $H^q(N, A)$ is really a G/N-module. Therefore the cohomology groups

$$H^p(G/N, H^q(N, A))$$

are defined for all $p, q \geqslant 0$. The Lyndon-Hochschild-Serre spectral sequence (or LHS spectral sequence for short) is a spectral sequence of bigraded groups

$$(E_2, d_2), (E_3, d_3), \ldots, (E_r, d_r), \ldots$$

such that

$$E_2^{p,q} = \begin{cases} H^p(G/N, H^q(N, A) & \text{if } p, q \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

and such that the differential d_r has bidegree $(r, 1 - r)$ for $r = 2, 3, 4, \ldots$. (For this spectral sequence, as is often the case, the first term is that for which $r = 2$.)

Unfortunately, the various methods of constructing this spectral sequence are all rather lengthy, and therefore cannot be described in a short essay like this.

While these conditions make clear that the cohomology of the groups N and G/N enters into the picture, it remains to explain the connection with the cohomology of G.

First of all, since $E_2^{p,q} = 0$ if either p or q is negative, it follows that for such values of p and q, $E_r^{p,q} = 0$ for *all* $r > 2$. Now consider the following groups and homomorphisms from the rth term of this spectral sequence

$$E_r^{p-r,q+r-1} \underset{d_r}{\to} E_r^{p,q} \underset{d_r}{\to} E_r^{p+r,q-r+1}.$$

Assume that $r > \max(p, q+1)$; this condition implies that

$$E_r^{p-r,q+r-1} = 0 \quad \text{and} \quad E_r^{p+r,q-r+1} = 0.$$

Hence, $Z_r^{p,q} = E_r^{p,q}$, and $B_r^{p,q} = 0$. It follows that

$$E_{r+1}^{p,q} = H^{p,q}(E_r, d_r) = Z_r^{p,q}/B_r^{p,q} = E_r^{p,q}.$$

In other words, *for fixed* (p, q), *the groups* $E_r^{p,q}$ *are the same for all sufficiently large values of* r. It is convenient to use the notation $E_\infty^{p,q}$ to denote the group $E_r^{p,q}$ for these large values of r.

We can now describe the connection of the spectral sequence with the cohomology groups of G. *For each* $n \geqslant 0$, *there is defined a nested sequence of subgroups of* $H^n(G, A)$,

$$H^n(G, A) = F^0H^n \supset F^1H^n \supset \cdots \supset F^nH^n \supset F^{n+1}H^n = 0,$$

such that for all $p, q \geqslant 0$,

$$E_\infty^{p,q} = F^pH^{p+q}/F^{p+1}H^{p+q}.$$

As is so often the case in mathematics, a special jargon has arisen to describe this situation. The nested sequence of subgroups of $H^n(G, A)$ is called a *filtration* of $H^n(G, A)$. In this case, the filtration is *decreasing*, since $F^kH^n \supset F^{k+1}H^n$. The doubly indexed sequence of quotient groups

$$F^pH^{p+q}/F^{p+1}H^{p+q}$$

is called the *associated bigraded group* of the graded group $\{H^n(G, A)\}$ with respect to the given filtration. The fact that

$$E_\infty^{p,q} = F^pH^{p+q}/F^{p+1}H^{p+q}$$

is expressed by saying that "the spectral sequence (E_r, d_r) converges to the associated bigraded group of $H^*(G, A)$ appropriately filtered." It should be pointed out, however, that the word "converges" is rather misleading in this context, since there is no limiting procedure involved. For a fixed (p, q), the groups $E_r^{p,q}$ reach their "limiting value" $E_\infty^{p,q}$ after only a finite number of steps.

Using this jargon, we can summarize all this as follows.

THEOREM. *Let N be a normal subgroup of the group G, and let A be a G-module. Then there exists a spectral sequence of bigraded groups*

$$(E_r, d_r), \qquad r = 2, 3, 4, \ldots,$$

such that

$$E_2^{p,q} = \begin{cases} H^p(G/N, H^q(N, A)) & \text{if } p, q \geqslant 0, \\ 0 & \text{otherwise}. \end{cases}$$

and such that d_r has bidegree $(r, 1-r)$. This spectral sequence converges to the associated bigraded group of $H^*(G, A)$ appropriately filtered.

At this stage, it probably seems plausible to the reader that this spectral sequence does indeed prescribe certain relations between the cohomology groups of N, G/N, and G; however, the whole thing is so complicated, and there is so much arbitrariness in this description, that it is difficult to see how one could ever make use of this machinery. We will come back to this point after our next example.

Second example, from topology. This example concerns the cohomology groups of a fibre space. Given two topological spaces X and Y, we can form the product space $X \times Y$. The notion of "fibre space" is a generalization of the concept of a product space; indeed, in Russian mathematical literature they are sometimes called "skew products." Let S^1 denote the unit circle $x^2 + y^2 = 1$ in the (x, y) plane, and let I denote the closed unit interval $[0, 1]$. Then one can form the product space $S^1 \times I$, which is an annulus, and $S^1 \times S^1$, which is a torus. But one can also slightly modify the construction, and by putting a "twist" in each of these spaces, form a Möbius strip and a Klein bottle respectively. These are the simplest nontrivial examples of a fibre space: locally, a Möbius strip is like an annulus, and a Klein bottle like a torus, but globally they are different.

The precise definition goes as follows: A *fibre space* is a quadruple (E, p, B, F), where E, B, and F are topological spaces and p is a continuous map of E onto B which satisfies the following condition: Given any point b of B, there exists a neighborhood U of b and a homeomorphism h of $U \times F$ onto $p^{-1}(U)$ such that

$$p[h(x, y)] = x$$

for any $(x, y) \in U \times F$.

This condition means that locally the map p is like the projection of a product space onto one of its factors, even though this may not be true globally. In one of the examples cited above, $E =$ the Möbius strip, $B = S^1$ and $F = I$; it should be clear to the reader how to define the continuous map p so that the required local product condition holds.

The spaces E, B, and F are customarily called the "total space," "base space," and "fibre" respectively, and the mapping p is called the "projection." The subspaces $p^{-1}(b)$ for any $b \in B$ are also called "fibres"; each of them is homeomorphic to F. In the last fifty years or so the fibre space concept has played an increasingly important role in algebraic topology itself, in related disciplines such as differential geometry and global analysis, and even in algebraic geometry. The basic definition given above has been modified, generalized, and specialized in almost every conceivable way.

An obvious question which arises in the study of fibre spaces is the following: Do the homology (or cohomology) groups of the base space and fibre determine the homology (or cohomology) groups of the total space? In the case of a product space, the answer is Yes: The homology groups of $X \times Y$ are completely determined by those of X and of Y. (This result is called the "Künneth Theorem.") However, with given choices for the base space and fibre, it is usually possible to construct many different fibre spaces (provided the base space is not contractible). Thus it seems likely that the answer to this question will be No (in general); more information is needed. It is perhaps more sensible to study the following problem: Determine what relations must exist between the homology (or cohomology) groups of the base space, fibre, and total space of a fibre space. The answer to this problem is given by another spectral sequence, called the *Leray-Serre spectral sequence*. The version for cohomology goes as follows:

THEOREM. *Let* (E, p, B, F) *be a fibre space with all three spaces arcwise connected and* B *simply connected, and let* A *be an abelian group (the coefficient group). Then there exists a spectral sequence of bigraded groups*

$$(E_r, d_r), \qquad r = 2, 3, 4, \ldots,$$

such that

$$E_2^{p,q} = \begin{cases} H^p(B, H^q(F, A)) & \text{if } p, q \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

and such that d_r *has bidegree* $(r, 1 - r)$. *The spectral sequence converges to the associated bigraded group of* $H^*(E, A)$ *appropriately filtered.*

EXPLANATORY REMARKS. (1) If X is a topological space and A is an abelian group, then $H^n(X, A)$ denotes the n-dimensional singular cohomology group of X with coefficient group A.

(2) The hypothesis that B is simply connected is unnecessary; however, if it is omitted, the description of $E_2^{p,q}$ involves what is known as "cohomology with local coefficients," and hence is more complicated.

(3) The reader will undoubtedly note a strong analogy between the Lyndon-Hochschild-Serre spectral sequence and the Leray-Serre spectral sequence. This analogy is not just an accident. By regarding the cohomology of a group as the cohomology of the corresponding Eilenberg-Mac Lane space, it is possible to "realize" the LHS spectral sequence as a particular case of the Leray-Serre spectral sequence.

How does one actually use a spectral sequence to get results? As was suggested previously, in general the LHS or Leray-Serre spectral sequence is very complicated, and it is difficult to extract much information out of it. However, if special hypotheses hold, it is often possible to get significant results. Here are some examples.

(1) Note that if $d_r = 0$, then $E_{r+1} = E_r$; hence if $d_r = 0$ for all $r \geqslant 2$, then $E_2 = E_\infty$. In this case the spectral sequence is said to "collapse." Whenever the LHS spectral sequence collapses, it is much easier to derive significant relationships between the cohomology groups of G, N, and G/N. (An analogous statement is true for the Leray-Serre spectral sequence.)

It may happen that enough of the groups $E_2^{p,q}$ are 0 so that the spectral sequence must of necessity collapse. For example, observe that each of the differentials d_r increases the "total degree", $p + q$, by one; hence it follows that if the groups $E_2^{p,q}$ which are nonzero all have even total degree (or they all have odd total degree), then $d_r = 0$ for all r, and the spectral sequence collapses.

It can also happen that a spectral sequence collapses for other reasons. For example, the Leray-Serre spectral sequence collapses if the coefficient group A is a field, and the fibre is "totally nonhomologous to zero," which is a shorthand way of saying that the inclusion map of a fibre into the total space E induces a *surjection* of the cohomology groups of E onto the cohomology groups of the fibre (with coefficients in the field A).

(2) Suppose that for some fixed integer $n \geqslant 1$, $E_2^{p,q} = 0$ unless $q = 0$ or $q = n$. Then the information contained in the spectral sequence is the same as that contained in a certain long exact sequence of abelian groups, and hence is relatively easy to extract. This case actually occurs for the Leray-Serre spectral sequence if the fibre F is an n-dimensional sphere, or at least has the same homology groups as an n-dimensional sphere. The resulting long exact sequence is called the Gysin sequence. Similar remarks are true if $E_2^{p,q} = 0$ except for $p = 0$ or $p = n$. This case occurs if the base space B is an n-dimensional sphere; the resulting long exact sequence is called the Wang sequence.

(3) One can be less ambitious and demanding in the amount of information one tries to extract from a spectral sequence. For example, assume that (E, p, B, F) is a fibre space such that the Leray-Serre spectral sequence is defined, as described above, and that the Euler characteristics of the spaces B and F are both defined (the Euler characteristic is the alternating sum of the ranks of the integral cohomology groups, provided this alternating sum is finite). Then the Euler characteristic of the total space E is also defined, and is equal to the product of the Euler characteristics of B and F. This theorem is easily proved using the Leray-Serre spectral sequence.

(4) Although spectral sequences are not as useful as one would hope, for making calculations in the specific cases, they are often valuable tools for proving rather general theorems. As an example of such a theorem, we present the following: Assume that (E, p, B, F) and (E', p', B', F') are fibre spaces which satisfy the hypotheses given above for the Leray-Serre spectral sequence. Assume that we have given a pair of continuous maps $f: E \to E'$ and $g: B \to B'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow p & & \downarrow p' \\ B & \xrightarrow[g]{} & B' \end{array}$$

is commutative. This hypothesis implies that f maps each fibre of the first fibre space into a fibre of the second (hence it is often called a "fibre-preserving" map).

It follows that there are induced homomorphisms of homology groups,

$$f_* : H_p(E, A) \to H_p(E', A),$$

$$g_* : H_q(B, A) \to H_q(B', A),$$

$$(f|F)_* : H_r(F, A) \to H_r(F', A),$$

which are defined for all integers p, q, and $r \geqslant 0$.

THEOREM. *Assume the coefficient group A is infinite cyclic. Then any two of the following conditions imply the third*:

(a) f_* *is an isomorphism for all* $p \geqslant 0$.

(b) g_* *is an isomorphism for all* $q \geqslant 0$.

(c) $(f|F)_*$ *is an isomorphism for all* $r \geqslant 0$.

This theorem is proved using the Leray-Serre spectral sequence for the homology of fibre spaces, which is very similar to the one described above for the cohomology of fibre spaces. One also uses what is called a "comparison theorem" for spectral sequences.

(5) Spectral sequences can often be given additional natural structures, which makes it possible to derive additional conclusions. For example, the LHS and Leray-Serre spectral sequences as described above are spectral sequences of bigraded groups, (E_r, d_r). However, if we assume that the coefficient group A is a commutative ring with unit, then it is possible to define *in a natural way* products in each of these bigraded groups, so they become bigraded rings. The differentials d_r have very nice properties vis-a-vis these products:

$$d_r(x \cdot y) = (d_r x) \cdot y \pm x \cdot (d_r y).$$

These products are frequently called "cup products."

History of the early development of spectral sequences. Spectral sequences were developed and first used in France immediately after World War II. Apparently they were first introduced by J. Leray in some papers concerned with what would nowadays be called the sheaf-theoretic cohomology of locally compact spaces; see [3] and the references given there. Soon afterwards they were being used by H. Cartan and J.-L. Koszul. (The latter used them in his work on the cohomology of Lie algebras.) The reviewer recalls quite vividly the difficulties algebraic topologists on this side of the Atlantic had in trying to understand and digest these first papers about spectral sequences. Fortunately, the famous theses of J.-P. Serre [**6**] and A. Borel [**1**] appeared within a few years. These theses gave many interesting new results which were derived using spectral sequences, and the exposition was exceptionally clear. (Serre and Borel both had announced the principal results of their theses in brief *Comptes Rendus* notes which appeared a year or so before their respective theses.)

Probably the first application of spectral sequences to algebra was the above-mentioned work of Koszul on Lie algebras. The first explicit statement of the LHS spectral sequence was in a *Comptes Rendus* note by Serre in 1950 [**5**], followed by a complete exposition by Serre and Hochschild in 1953 [**2**]. The earlier work of R. C. Lyndon, contained in his 1946 Harvard University Ph.D. thesis and published in 1948 [**4**], was done before spectral sequences were known in the U.S.; hence he had to try to state his results without the use of spectral sequences.

After the publication of the theses of Serre and Borel, spectral sequences techniques gradually pervaded more and more of algebraic topology and homological algebra. They also came to be used in certain other parts of mathematics, such as algebraic geometry, category theory, and algebraic

K-theory. To the best of the reviewer's knowledge, nobody has ever tried to catalog all the many different applications of spectral sequences in modern-day mathematics.

The book by McCleary. Other than the original papers, most expositions of spectral sequences have been in textbooks on some aspect of algebraic topology or homological algebra, and are aimed at a few particular applications. From a pedagogical point of view, this is most sensible, because the only way one can possibly learn much about spectral sequences is to actually work with them and get used to doing all the gory details. A reader of the introduction to McCleary's new book might be misled into thinking it is a general, abstract book about spectral sequences, equally applicable to whatever field of mathematics they might be applied to. Fortunately McCleary has not tried to write that kind of book. What he has actually written is a guide to the use of spectral sequences in algebraic topology. Only one brief chapter (10 pages out of the books' total of 423 pages) is devoted to applications of spectral sequences in algebra, algebraic geometry, etc.

As a matter of fact, the major part of the book is concerned with four particular spectral sequences in algebraic topology: the Leray-Serre spectral sequence, the two Eilenberg-Moore spectral sequences, and the Adams spectral sequence of stable homotopy theory. Undoubtedly most algebraic topologists would agree that these four spectral sequences are the most important ones in the subject. The first three of these are applications to fibre spaces, hence the expositions of these three fit together rather nicely.

The author states in the introduction that a student of algebraic topology who studies this book should have a basic knowledge of singular homology theory, homotopy groups, and homological algebra. Chapter 4, which is almost 40 pages long and is entitled "Topological Background," reviews rather briefly a number of more advanced concepts that are needed in the book.

But even this knowledge would probably not be enough to understand some of the sections which are specifically labeled "not for the novice." Thus this book is both a textbook for beginners in the subject and an encyclopedic reference book for experts. Its value as a reference book is enhanced by a lengthy bibliography. However, in spite of its encyclopedic nature, the author very wisely does not try to give all the details of every proof; for example, regarding the introduction of cup products into the Leray-Serre spectral sequence, the author states the main results, and refers the reader to Serre's thesis and a couple of other references for the details.

In an encyclopedic book about a subject as complicated as spectral sequences, it is inevitable that any author would inadvertently introduce a fair number of errors. This reviewer did not notice very many obvious errors; most of them will only be detected by extremely careful study. Presumably these will be caught by alert readers and can be corrected in subsequent printings.

McCleary has undertaken and completed a daunting task; few algebraic topologists would have the courage to even try to write a book such as this. The mathematical community is indebted to him for this achievement!

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Prescribing the curvature of a Riemannian manifold, by Jerry L. Kazdan, CBMS Regional Conference Series in Mathematics, vol. 57, American Mathematical Society, Providence, R. I., 1985, vii + 55 pp., $12.00. ISBN 0-8218-0707-2

It is generally believed that a most interesting area in nonlinear partial differential equations lies in the study of special equations, particularly those arising from geometry and physics. The monograph under review is an account of problems on the existence of a Riemannian metric with given curvature conditions. It contains some of the most important results in mathematics in recent years.

There are all kinds of curvatures: Gaussian curvature, scalar curvature, Ricci curvature, Riemannian sectional curvature, etc. When any one of them is prescribed, we get a system of partial differential equations on the fundamental tensor of the Riemannian metric. The problems have a meaning for a manifold without boundary, giving rise to some problems attractive because of their simplicity. But Chapter IV of these notes gives a treatment of some recent developments on boundary-value problems.

Even for the Gaussian curvature there are unanswered questions. To be the Gaussian curvature of a compact surface M^2, a function $K \in C^\infty(M^2)$ must satisfy a sign condition forced by the Gauss-Bonnet theorem. Kazdan and Warner proved that this is sufficient. It would be interesting to prove this by a conformal transformation of a given metric g_0 on M^2. There will be no difficulty if the Gaussian curvature K_0 of g_0 is negative. For $K_0 > 0$ there are further necessary conditions and it is not known whether they are sufficient. Even for the two-sphere $M^2 = S^2$ it is not known whether one could obtain a metric of constant curvature through the conformal transformation of a given one.

The simplest generalization of the Gaussian curvature to higher dimensions is the scalar curvature, which is a scalar invariant. By applying the Bochner