

ON THE LIE SUBGROUPS OF INFINITE DIMENSIONAL LIE GROUPS

J. LESLIE

1. Introduction. Milnor [4] posed the question, "Does every closed Lie subalgebra of the Lie algebra of an infinite-dimensional Lie group modelled on a complete locally convex topological vector space correspond to an immersed Lie subgroup?" Omori [5] has shown that the answer to this question is negative in general; in this note we outline conditions under which a positive answer can be given.

Correspondence with Milnor has been helpful in preparing the present version of this research announcement.

All of the theorems in this paper have proofs which are written down. The author is in the process of preparing a manuscript in which detailed proofs are presented.

Let us recall that a subset of a real vector space E is said to absorb a subset B of E when there exists a constant $\lambda > 0$ so that $\lambda B \subseteq A$.

We shall call a subset, S , of a vector space which is circled (i.e. $|\lambda| \leq 1$ and $s \in S$ implies $\lambda s \in S$) and convex a disk.

DEFINITION 1. A bornological vector space is a Hausdorff, locally convex, topological vector space in which any disk which absorbs every bounded subset of E is a neighborhood of the origin.

EXAMPLE 1. A metrizable locally convex topological vector space is bornological [1].

EXAMPLE 2. The locally convex inductive limit of bornological spaces is bornological [1].

CONVENTION. C° will mean continuous in this paper.

DEFINITION 2. We say that $f: U \rightarrow F$ is Gateaux C^n smooth when there exists k -multilinear symmetric continuous functions $D^k f(x): E \times \cdots \times E \rightarrow F$, $1 \leq k \leq n$, $x \in U$, so that each

$$D^k f: U \times E \times \cdots \times E \rightarrow F$$

is continuous and each

$$F_k(v) = f(x+v) - f(x) - Df(x)(v) - \cdots - \frac{1}{k!} D^k f(x)(v, \dots, v), \quad 1 \leq k \leq v,$$

satisfies the property that

$$G_k(t, v) = \begin{cases} F_k(tv)/t^k & t \neq 0, \\ 0, & t = 0, \end{cases}$$

is continuous at $(0, v)$.

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DEFINITION 3. Given a Lie group G modelled on a complete bornological space \mathcal{G} the manifold structure on G gives rise to a local trivialization of the tangent bundle TG at $e \in G$ over a coordinate neighborhood of G at e , say U , so that $TU \approx U \times \mathcal{G}$. In general a right invariant vector field ξ will define with respect to this trivialization a nonconstant function $X_\xi: U \rightarrow \mathcal{G}$. We say that the Lie group is *nice* when given any pair of closed, bounded disks $B, C \subseteq \mathcal{G}$ there exists a sequence of closed, bounded disks C_1, \dots, C_n, \dots and $0 < \varepsilon < 1$ so that

- (1) $X_\xi(x_0 + \varepsilon C) \subseteq C_1$, for $\xi \in \varepsilon B$.
- (2) $x_0 + \varepsilon C + (\varepsilon/1!)C_1 + \dots + (\varepsilon^n/n!)C_n \subseteq U$.
- (3) There exists a positive integer p and a closed bounded disk D so that $D_n = \sum_{q \geq n} (1/q!)C_q \subseteq D$, for $n \geq p$, converges to 0 in F for the D -gauge norm topology on F_D .

DEFINITION 4. Let E be a topological vector space and $U \subseteq E$ a neighborhood of the origin, a U -system of generators \mathcal{B} of the bounded sets of E is a collection of bounded subsets of U so that

- (i) given $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ with $B_1 \cup B_2 \subseteq B_3$;
- (ii) given any bounded subset $C \subseteq E$, there exists $B \in \mathcal{B}$ which absorbs C .

DEFINITION 5. A nice Lie group is called perfect when with respect to the canonical coordinate system (U, ϕ) of the definition of niceness (see Definition 3) we have a neighborhood U_0 of the identity with $U_0^2 \subseteq U$ so that $\phi(U_0) \subseteq \mathcal{G}$ is a convex open neighborhood of the origin and so that given any pair, $a \in U_0$ and $0 < l_a < 1$, we have that there exists a U_0 -system of generators \mathcal{B} so that $B \in \mathcal{B}$, and $h \in B$ implies

$$(D_x(\phi \circ R_a \circ \phi^{-1})_h - D_x(\phi \circ R_a \circ \phi^{-1})_0)(B) \subseteq l_a D_x(\phi \circ R_a \circ \phi^{-1})_0(B);$$

for all $B \in \mathcal{B}$, where $D_x(\)_y$ is differentiation at y .

EXAMPLES. (a) Banach-Lie groups are perfect.

(b) Let $G_n \subseteq G_{n+1}$ be a sequence of Banach-Lie groups. Then $\varinjlim_n G_n$ is a perfect Lie group.

(c) Let M be a compact C^∞ manifold without boundary, then $\text{Diff}^\infty(M)$ with the C^∞ topology is perfect.

(d) Let M be a compact real analytic manifold without boundary, then $\text{Diff}^\omega(M)$ with C^ω topology is perfect.

(e) Let V_1 be a finite-dimensional vector space over R and let $V = V_1 + \dots + V_i + \dots$ be the free Lie algebra generated by V . The Campbell-Hausdorff formula defines a perfect Lie group structure on $\bar{V} = \prod_{i \in \mathbb{N}} V_i$ with the Cartesian product topology.

2. Statement of Theorem.

DEFINITION 6. Given a Lie group G with Lie algebra \mathcal{G} . A closed Lie subalgebra \mathcal{K} which has a closed complement \mathcal{M} in \mathcal{G} is called *admissible* when there exists a chart at $e \in G$, (ϕ, U) , and a direct sum decomposition $\mathcal{G} = \mathcal{M} + \mathcal{K}$ so that the composition

$$\psi_x: \xi \rightarrow \pi_{\mathcal{M}} \circ X_\xi(x), \quad \xi \in \mathcal{M},$$

is an isomorphism onto \mathcal{H} for each $x \in U$, where $\pi_{\mathcal{H}}: \mathcal{G} = \mathcal{H} + \mathcal{K} \rightarrow \mathcal{H}$ is the canonical projection; further we suppose that the map

$$\Phi: U \rightarrow L(\mathcal{H}, \mathcal{H})$$

given by $x \rightarrow \psi_x^{-1}$ is locally strongly bounded (i.e. given B a bounded disk in \mathcal{H} and $x_0 \in U$, there exists $\varepsilon > 0$ so that $\Phi(x_0 + \varepsilon B)(C) \subseteq \mathcal{H}$ is bounded for every bounded $B \subseteq \mathcal{H}$).

THEOREM. *Let G be a perfect Lie group modelled on a complete bornological space with Lie algebra \mathcal{G} , and suppose that \mathcal{H} is an admissible Lie subalgebra of \mathcal{G} . Then there exists a Lie subgroup H having \mathcal{H} as its Lie algebra.*

EXAMPLES. (a) The C^∞ divergence free vector fields and the C^∞ locally Hamiltonian vector fields form admissible subalgebras of the Lie algebra of $\text{Diff}^\infty(M)$, where M is a compact C^∞ manifold without boundary.

(b) Banachable Lie subalgebras with a closed complement in the Lie algebra of a nice Lie group are admissible.

The verification of Example (a) depends on the so-called normal coordinates of Omori [6].

3. Remarks on the proof. The principal tool in the proof of the above theorem is the following version of a Frobenius theorem. This is a subtle theorem, as in the classical case it depends on an existence and uniqueness argument and on the smooth dependence on initial conditions.

THEOREM. *Let E and F be complete bornological spaces and $E' \subseteq E$ and $F' \subseteq F$ open sets. Suppose $f: E' \times F' \times E \rightarrow F$ is a C^2 map linear in E . Suppose for each $(x, y) \in E' \times F'$ and each pair $(a, b) \in E \times E$ that the map*

$$(a, b) \rightarrow \frac{\partial f}{\partial x}(x, y, a; b) + \frac{\partial f}{\partial y}(x, y, a; f(x, y, b))$$

is symmetric in (a, b) .

Further, suppose that given $x_0 \in E'$, $y_0 \in F'$ and closed, bounded disks $B \subseteq E$ and $C \subseteq F$ that there exists $0 < \varepsilon < 1$ and a sequence of closed, bounded disks $C_n \subseteq F$ so that $x_0 + \varepsilon B \subseteq E'$, $y_0 + \varepsilon C \subseteq F'$ and so that

$$(1) f(x_0 + \varepsilon B, y_0 + \varepsilon C, B) \subseteq C_1;$$

$$(2)$$

$$y_0 + \varepsilon C + \frac{\varepsilon}{1!}C_1 + \dots + \frac{\varepsilon^n}{n!}C_n \subseteq F'$$

and

$$\frac{\partial f}{\partial y}(x_0 + \varepsilon B, y_0 + \varepsilon C + \frac{\varepsilon}{1!}C_1 + \dots + \frac{\varepsilon^n}{n!}C_n)(C_n) \subseteq C_{n+1};$$

(3) there exists a positive integer p and a closed, bounded disk D , so that

$$D_n = \sum_{q \geq n} \frac{\varepsilon^q}{q!}C_q \subseteq D$$

for $n \geq p$ converges to 0 in $F_D = \bigcup_{\lambda \geq 0} \lambda D$ for the D gauge norm topology. Then there exist open neighborhoods U_0 of x_0 in E_1 and V_0 of y_0 in F_1 and a unique C^2 -mapping

$$\alpha: U_0 \times V_0 \rightarrow F$$

so that $\alpha(x_0, y) = y$ and $D_1 \alpha(x, y) = f(x, \alpha(x, y))$.

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON,
ILLINOIS 60201