11. M. A. Pinsky, *Averaging an alternating series*, Math. Magazine **51** (1978), 235–237.
12. D. F. Riddle, *Calculus and analytic geometry* (2nd ed.), Wadsworth, Belmont, California, 1974.
13. J. R. Schmidt, Philos. Mag. **32** (1941), 369–383.
14. D. Shanks, *Non-linear transformations of divergent and slowly convergent sequences*, J. Math. Phys. **34** (1955), 1–42.
15. W. F. Trench, *An algorithm for the inversion of finite Hankel matrices*, Siam. J. Appl. Math. **13** (1965), 1102–1107.
16. R. R. Tucker, Thesis, Oregon State University, 1963.
17. ______, *The δ^2-process and related topics*, Pacific J. Math. **22** (1967), 349–359.
18. ______, *The δ^2-process and related topics*. II, Pacific J. Math. **28** (1969), 455–463.
19. ______, *A general theory of alternating series*, 1986 (in preparation).

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Linear turning point theory, by Wolfgang Wasow, Applied Mathematical Sciences, vol. 54, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1985, ix + 246 pp., $38.00. ISBN 0-387-96046-5

The transitions of solutions of a linear differential equation from oscillatory to exponentially growing or exponentially decaying behavior as the independent variable, for example, changes sign are phenomena of interest to physicists and other scientists, primarily in the past sixty years, and continuing even today. The simplest equation exhibiting such behavior is Airy's equation

$$\text{(A)} \qquad y'' + xy = 0,$$

obtained from his study of the rainbow [2]. If we set $x = [\psi(0)\varepsilon^{-2}]^{1/3}t$, then the rainbow equation (A) becomes

$$\text{(A*)} \qquad \varepsilon^2 d^2y/dt^2 + t\psi(0)y = 0,$$

which might well be expected to have solutions close to solutions of the equation

$$\text{(A\#)} \qquad \varepsilon^2 d^2y/dt^2 + t\psi(t)y = 0 \qquad (\psi(0) \neq 0).$$

This idea occured to R. Gans in 1915 [6] in his investigations of total reflection in physical—as opposed to geometrical—optics. The point $t = 0$ is called a (simple) turning point, one where solutions of (A#) change from oscillatory to exponential behavior. An obvious mathematical question, only answered much later by R. E. Langer [8, 9] and others, is whether one can find changes of variables in (A) such that (A#) becomes (A*) with $\psi(0) = 1$ and with a small error term included. If the transformation from (A#) to A(*) is exact, it turns out that often it is but a formal power series in ε with coefficients holomorphic in the complex variable x, i.e., an asymptotic series which converges only for

$\varepsilon = 0$. The construction of such formal series has connections with topology and algebra (see V. I. Arnol'd [**3**]) and is related to problems in celestial mechanics and existence of normal forms in bifurcation theory; see Arnol'd [**4**, Chapter 5] and Hassard, Kazarinoff and Wan [**7**, Chapter 1 and Appendix 1].

When one deals with linear systems of differential equations, the situation is much more complex, and there are open problems in abundance. In linear systems new phenomena appear: transformations reducing a system to "simpler" form can introduce many new turning points and thus complicate, rather than simplify, analysis in the large.

Wasow's book is a mathematician's view of this important subject. It is the only such book. Y. Sibuya's monograph [**11**] treats only the important, and highly difficult, class of linear second-order equations with polynomial coefficients. The author's earlier book [**12**] is much broader in focus, and is out of date with respect to the problems treated in the book under review. For any graduate student who wants to do research in linear turning point theory, this book is a treasure house of methods and open problems. The specialist will also find much of interest. This reviewer, for example, learned of Gans' early contribution from Wasow's comprehensive Historical Introduction. Chapter IX on Fëdoryuk's global theory of second-order equations [**5**] contains new, nontrivial and interesting contributions by the author. Existence of a solution of

$$\text{(B)} \qquad \begin{gathered} \varepsilon u'' + f(x,\varepsilon)u' + g(x,\varepsilon)u = 0 \quad \text{on } [-1,1] \text{ with} \\ u(0,\varepsilon) = A, \qquad u(1,\varepsilon) = B \end{gathered}$$

that converges uniformly to a bounded nonzero function on $[-1,1]$ as $\varepsilon \to 0$ has been called *asymptotic resonance* by Ackerberg and O'Malley [**1**]. In Chapter XII, Wasow establishes new sufficient conditions for asymptotic resonance in the case where the differential equation in (B) has a simple turning point $f(x,0) = -xh(x)$, $h(x) > 0$ on $[-1,1]$). Little is known when (B) involves a turning point of higher order.

Your reviewer expresses but one dissatisfaction: specific applications to physical problems are omitted. Fortunately there are sources in which these can be found; for example, C. C. Lin's book on hydrodynamic stability [**10**].

The fruits of the work of Henri Poincaré are still ripening. *Linear turning point theory* is a particularly attractive one in taste and quality.

References

1. R. C. Ackerberg and R. E. O'Malley, *Boundary layer problems exhibiting resonance*, Studies in Appl. Math. **49(1970)**, 277–295.

2. G. B. Airy, *On the intensity of light in the neighborhood of a caustic*, Trans. Cambridge Philos. Soc. **6** (1838), 379–402.

3. V. I. Arnol'd, *On matrices depending on parameters*, Uspehi Mat. Nauk **26**, No. 2(1971), 101–114; English transl. in Russian Math. Surveys 26, No. 2(1971), 29–43.

4. ______, *Supplementary chapters of the theory of ordinary differential equations*, "Nauka", Moscow, 1978, 304 pp. (Russian)

5. M. A. Evgrafov and M. V. Fëdoryuk, *Asymptotic behavior as* $\lambda \to 0$ *of the solution of the equation* $w''(z) - p(z,\lambda)w(z) = 0$ *in the complex z-plane*, Uspehi Mat. Nauk **21**, No. 1 (127) (1966), 3–50. English transl. in Russian Math. Surveys **21** (1966), 1–48.

6. R. Gans, *Fortpflanzung des Lichts durch ein inhomogenes Medium*, Ann. Phys. **47** (1915), 709–736.

7. B. D. Hassard, N. D. Kazarinoff and Y.-H. Wan, *Theory and applications of Hopf bifurcation*, London Math. Soc. Lecture Series, No. 41, Cambridge Univ. Press, Cambridge, London, New York, 1981.

8. R. E. Langer, *On the asymptotic solutions of ordinary differential equations with an application to the Bessel functions of large complex order*, Trans. Amer. Math. Soc. **34** (1932), 447–480.

9. ______, *The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to the Stokes phenomenon*, Bull. Amer. Math. Soc. **40** (1934), 545–582.

10. C. C. Lin, *The theory of hydrodynamic stability*, Cambridge Univ. Press, Cambridge, 1966.

11. Y. Sibuya, *Global theory of a second order linear differential equation with a polynomial coefficient*, North-Holland Math. Studies no. 18, North-Holland-American Elsevier Publ. Co., Amsterdam-New York, 1975.

12. W. Wasow, *Asymptotic expansions for ordinary differential equations*, Interscience Publ., New York, 1965.

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Theory of multipliers in spaces of differentiable functions, by V. G. Maz'ya and T. O. Shaposhnikova, Monographs and Studies in Mathematics, vol. 23, Pitman Publishing Co., Brooklyn, New York, 1985, xii + 344 pp., $49.95. ISBN 0-273-08638-3

1. Multipliers. One of the simplest examples of a multiplier in a space of differentiable functions is a measurable function $\gamma(x)$, $x \in \mathbf{R}^n$, such that the operator of pointwise multiplication $u \to \gamma \cdot u$ is bounded from the Sobolev space W_2^1 on $\mathbf{R}^n$ into L_2 on $\mathbf{R}^n$; equivalently, there is a constant c such that

$$(1) \qquad \int |\gamma(x)\cdot\phi(x)|^2\,dx \leqslant c\int \left(|\nabla\phi(x)|^2 + |\phi(x)|^2\right)dx$$

for all $\phi \in C_0^\infty(\mathbf{R}^n)$. The space of all such γ is denoted by $M(W_2^1 \to L_2)$, with the smallest c in (1) the square of the multiplier norm of γ. Clearly, one can easily extend this notion to pairs of higher-order Sobolev spaces: $W_p^m \to W_q^k$, $k \leqslant m$, $1 \leqslant p, q < \infty$, or for that matter, to any of the various pairs of function spaces that naturally occur in analysis. The coefficients of a differential operator acting on Sobolev functions can be interpreted as multipliers. For example, if $P(x, D)u = \sum_{|\alpha|\leqslant k} a_\alpha(x) D_x^\alpha u$, then P: $W_p^m \to W_p^{m-k}$ is continuous when $a_\alpha \in M(W_p^{m-|\alpha|} \to W_p^{m-k})$. The function γ is called a compact multiplier if the operator of pointwise multiplication is a compact operator. The principal theme of the book under review (referred to below as *Multipliers*) is the characterization of multipliers and compact multipliers in the basic Sobolev-type spaces used in analysis. Because of their connection to differential equations, it is not surprising that there are plenty of sufficient conditions in the literature for multipliers or compact multipliers. For example,