PROPAGATION OF SINGULARITIES OF SOLUTIONS
TO SEMILINEAR BOUNDARY VALUE PROBLEMS

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ABSTRACT. Let P be a second-order, strictly hyperbolic differential operator on an open region $\Omega \subset \mathbf{R}^n$ ($n \geq 3$) with smooth noncharacteristic boundary. Given a solution $u \in H^s_{\text{loc}}(\overline{\Omega})$, $s > (n+1)/2$, to $Pu = f(x, u)$, we discuss the propagation of microlocal H^r singularities in the range $s \leq r < 2s - n/2$ in the general case where the Hamilton field of p may be tangent to $\partial T^*\Omega\backslash 0$ to arbitrarily high finite or infinite order.

Introduction. Recently, M. Sablé-Tougeron [10, 11] and A. Alabidi [1] used the calculus of paradifferential operators to obtain, for a quite general class of nonlinear boundary value problems, results describing the reflection of singularities that travel on bicharacteristics transversal to the boundary of a region $\Omega \subset \mathbf{R}^n$. Using a different approach, similar to the one initiated by Rauch [9] and further developed by Beals and Reed [3] in studying interior propagation, we have obtained results for second-order semilinear problems with Dirichlet conditions in the general case where tangential bicharacteristics as well as gliding rays may carry singularities. Our argument has two main steps. We first prove an analogue of Rauch's Lemma for a class of spaces measuring microlocal H^s regularity up to the boundary. We also require a precise linear theorem (Theorem 2), a refinement of the results of Melrose and Sjöstrand ([8], see also [6]), which describes the propagation of H^s regularity along generalized bicharacteristics. This theorem must apply to equations $Pu = v$ for distributions v which lie in our microlocal algebras, but which cannot be assumed to be normally regular. A simple inductive argument combining the above then yields the desired semilinear theorem (Theorem 1). Complete proofs will appear in [4].

Spaces of distributions near $\partial\Omega$. Let $\tilde{U}$ be a coordinate chart centered at $x_0 \in \partial\Omega$, and let (x_1, x') be coordinates in which $\overline{\Omega} \cap \tilde{U}$ ($\equiv U$) = $\{x_1 \geq 0\}$. We will use the spaces $H^{\text{loc}}_{t,t'}(U)$ defined in Hörmander [5]. For $u \in H^{\text{loc}}_{r,-\infty}(U)$ we define $\widetilde{WF}_r(u) \subset (T^*\partial U\backslash 0) \cup (T^*\overset{\circ}{U}\backslash 0)$ as follows. If $\sigma \in T^*\partial U\backslash 0$, $\sigma \notin \widetilde{WF}_r u$ if and only if for some tangential pseudodifferential operator $\phi(x, D')$ of order zero, elliptic at $(0, \sigma)$, $\phi u \in H^r_{\text{loc}}(U)$. In this case we write $u \in \tilde{H}^r(\sigma)$. In $T^*\overset{\circ}{U}$, $\widetilde{WF}_r$ coincides with the usual notion of WF_r. Note that the definition of $\widetilde{WF}_r u$ is coordinate dependent. However, it is a consequence (see [4]) of the Lemma stated below and the fact that $\partial\Omega$ is noncharacteristic, that for

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solutions $u \in H_{\mathrm{loc}}^s(\overline{\Omega})$, $s > (n+1)/2$, to $Pu = f(x, u)$, $\widetilde{WF}_r u$ is coordinate independent for $s \leq r < 2s - n/2 + 1/2$.

It was observed in [**10**] that the spaces $H_{t, t'}^{\mathrm{loc}}(U)$ are algebras stable under $f(x, u)$ if $t > \frac{1}{2}$, $t + t' > n/2$, and $t + 2t' > \frac{1}{2}$. Thus, given $u \in H_{\mathrm{loc}}^s$ such that $Pu = f(x, u)$, Peetre's Theorem [**5**, 4.3.1] yields by induction that $u \in H_{2s-\rho+2, -s+\rho-2}^{\mathrm{loc}}(U)$ for all $\rho > \frac{1}{2}$. This suggests considering, for $\sigma \in T^* \partial U \backslash 0$, the following spaces as candidates for microlocal algebras at the boundary:

$$\mathcal{A}_{s, s', \rho}(\sigma) = H_{2s-\rho, -s+\rho}^{\mathrm{loc}}(U) \cap \tilde{H}^{s'}(\sigma).$$

Main results. The following lemma is, we think, the natural analogue of Rauch's Lemma at the boundary.

LEMMA. *The spaces $\mathcal{A}_{s, s', \rho}(\sigma)$ defined above are algebras stable under the action of C^∞ functions $f(x, u)$ provided $(n+1)/2 < s \leq s' < 2s - n/2 - \rho$ and $\rho \in (1/2, s - n/2)$.*

In the statement of the following theorem we take $\Omega = \omega \times \mathbf{R}_{x_n}$, where $\omega \subset \mathbf{R}^{n-1}$ is an open region with smooth boundary. $\mathcal{H}$ and $\mathcal{G}$ denote the hyperbolic and glancing regions of $T^* \partial \Omega \backslash 0$ (see [**8** or **6**]).

THEOREM 1. *Let P be a second-order differential operator, noncharacteristic with respect to $\partial \Omega$, and strictly hyperbolic with respect to the surfaces $x_n = c$. Suppose that $u \in H_{\mathrm{loc}}^s(\overline{\Omega})$, $s > (n+1)/2$, satisfies $Pu = f(x, u)$ in Ω. Then for $s \leq r < 2s - n/2$, every point $\gamma_0 \in \widetilde{WF}_r u \backslash WF_r(u|_{\partial \Omega})$ is either a characteristic point of P in $T^* \Omega$, or else contained in $\mathcal{H} \cup \mathcal{G}$. An open interval $(-T, T) \ni t \to \gamma(t)$ with $\gamma(0) = \gamma_0$ on a generalized bicharacteristic [**8**, **6**] is contained in $\widetilde{WF}_r u$.*

Theorem 1 is proved by an inductive argument using the lemma and the following linear theorem. Note that the assumptions on P in Theorem 1 and the coordinate invariance of $\widetilde{WF}_r u$ for $u \in H_{\mathrm{loc}}^s$ satisfying $Pu = f(x, u)$ ($r \in [s, 2s - n/2 + \frac{1}{2})$) allow us to reduce to the case where P and Ω are as described in Theorem 2. We denote points in $T^* \mathbf{R}^n$ by (x_1, x', ξ_1, ξ').

THEOREM 2. *Suppose $\Omega \subset \mathbf{R}_+^n = \{x_1 > 0\}$ and write $\partial \Omega = \overline{\Omega} \cap \partial \mathbf{R}_+^n$. Let $P = D_{x_1}^2 - R(x, D')$ be a second-order differential operator on Ω strictly hyperbolic with respect to the planes $x_n = c$. Suppose $Pu = f$ in Ω, where for some $s' \geq 1$, $u \in H_{s'+2, -\infty}^{\mathrm{loc}}(\overline{\Omega})$, $f \in H_{s', -\infty}^{\mathrm{loc}}(\overline{\Omega})$, and $u \in H^{s'+1}\{(x, \xi_1, 0): x_1 > 0, x \text{ near } \partial \Omega\}$. Then every point $\gamma_0 \in \widetilde{WF}_{s'+1/2} u \backslash (\widetilde{WF}_{s'} f \cup WF_{s'+1/2}(u|_{\partial \Omega}))$ is either a characteristic point of P in $T^* \Omega$, or else contained in $\mathcal{H} \cup \mathcal{G}$. An interval $(-T, T) \ni t \to \gamma(t)$ with $\gamma(0) = \gamma_0$ on a generalized bicharacteristic is contained in $\widetilde{WF}_{s'+1/2} u$.*

Sketch of the proofs. We begin with the lemma. Any $u \in \mathcal{A}_{s, s', \rho}(\sigma)$ can be extended to an element $\tilde{u} \in H_{\mathrm{loc}}^s(\tilde{U})$ such that $\tilde{u} = u_1 + u_2$, where $u_1 \in H_{\mathrm{loc}}^{s'}(\tilde{U})$ and where $u_2 \in H_{2s-\rho, -s+\rho}^{\mathrm{loc}}(\tilde{U})$ has the property that $\phi(x, D')u_2 \in H_{2s-\rho, \infty}(\mathbf{R}^n)$ for $\phi(x, D')$ supported sufficiently near $(0, \sigma)$. Now given u and v in $\mathcal{A}_{s, s', \rho}(\sigma)$, we first choose such extensions $\tilde{u} = u_1 + u_2$, $\tilde{v} = v_1 + v_2$. We then show that tangential operators of order zero of the form $\chi(D') \circ \psi^2(x)$,

where $\psi \in C_0^\infty(\mathbf{R}^n)$, map $\tilde{u}\tilde{v}$ into $H^{s'}(\mathbf{R}^n)$ provided they are supported sufficiently near $(0,\sigma)$. For $u_1 v_1$ this is immediate, and the convolutions corresponding to the products $u_1 v_2$ (or $u_2 v_1$) and $u_2 v_2$ are examined separately. The stability of the algebras under smooth $f(x,u)$ then follows by considering $Df(x,u)$, using the chain rule and an inductive argument like the one given in Beals [**2**, Corollary 1.8].

Next we will sketch an argument that can be used to prove Theorem 2 when, for example, $\gamma_0 \in \mathcal{H}$ or $\gamma_0 \in \mathcal{G}_d$ (the diffractive region). Say $\gamma_0 = \sigma \in \mathcal{G}_d$, where the x_n-coordinate of σ is 0. Given $Pu = f(x,u) \in \tilde{H}^{s'}(\sigma)$, $u|_{\partial U} \in H^{s'+1/2}(\sigma)$, $u \in H^{s'+1/2}(\Gamma_-)$, where $\Gamma_- \subset T^*\overset{\circ}{U}$ is the portion in $x_n > -\varepsilon$ of the incoming tangential ray passing over σ, we must show $u \in \tilde{H}^{s'+1/2}(\sigma)$. We can suppose $f \in H^{s'}(\Gamma_-)$. By solving a mixed problem in U we first find a distribution $w \in H_{\text{loc}}^{s'+1/2}(U)$ such that $u' = u - w$ satisfies $Pu' = f$, $\sigma \notin WF$ $u'|_{\partial U}$, $u' \in H^{s'+1}(\Gamma_-)$. Extend f to an element $\tilde{f} \in H_{s',-\infty}^{\text{loc}}(\tilde{U})$ such that $\chi(x,D')\tilde{f} \in H^{s'}(\mathbf{R}^n)$ for all $\chi \in L^0$ supported sufficiently near $(0,\sigma)$. Now solve, in $\tilde{U}$, the extension problem:

$$Pv = \tilde{f}, \quad v = u' \quad \text{near } \Gamma_- \text{ in } x_n < -\varepsilon/2.$$

One then shows that $A(x,D')v \in H^{s'+1}(\mathbf{R}^n)$ for all $A \in L^0$ supported sufficiently near $(0,\sigma)$. (It is obvious that $Av \in H^{s'+1}(p^{-1}(0))$ for such A.) Next choose such an $A \equiv 1$ near $(0,\sigma)$ and solve the mixed problem in U:

$$Pw' = 0, \quad w'|_{\partial U} = A(u' - v)|_{\partial U}, \quad w' = 0 \text{ in } x_n < -\varepsilon/2.$$

Since $Av|_{\partial U} \in H^{s'+1/2}$, $w' \in H_{\text{loc}}^{s'+1/2}(U)$. The results of Melrose [**7**] and Taylor [**12**] imply $\sigma \notin WF_b(u' - (v + w'))$. Hence $u' \in \tilde{H}^{s'+1/2}(\sigma)$, which implies $u \in \tilde{H}^{s'+1/2}(\sigma)$ as desired.

In cases of finite-order tangency where the incoming or outgoing generalized bicharacteristic is a gliding ray, and in the case of infinite-order tangency, our proof of Theorem 2 follows the arguments in Hörmander [**6**, §24.5] closely. Again, an appropriate mixed problem is solved first. The remainder of the proof involves checking that the L^2 pairings in [**6**, 24.5] remain bounded under the hypotheses of Theorem 2.

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NOTE ADDED IN PROOF. The proof of Theorem 2 has been simplified in [**4**]. A method similar to that sketched above for $\sigma \in \mathcal{G}_d$ can be used to handle all points $\sigma \in \mathcal{H} \cup \mathcal{G}$. The revised argument allows H^r regularity for $r < 2s - n/2 + \frac{1}{2}$ to be propagated in Theorem 1.

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