to μ, and $\int x\,d\mu = \varphi(h_\mu)$, for every $\mu \in M_E$; (ii) there is a $\psi \in C''$ such that $\psi \upharpoonright E = \varphi \upharpoonright E$ and $\delta(\psi)(f) = 0$ for every $f \in E$; (iii) $\varphi \upharpoonright K_E$ is $\mathfrak{T}_s(C', C)$-continuous.

Let $Ra \subseteq C''$ be the set of those φ such that $u(|\varphi|)^*$ is zero except on a meagre set. Then Ra is a norm-closed solid linear subspace of C''; its polar in C' is precisely $C^\times$. (Note that $C^\times = \{0\}$ in many of the most important elementary cases.) The quotient C''/Ra is an M-space with unit; let $\tilde{C}$ be the image of $\hat{C}$ in C''/Ra; because $\hat{C} \cap Ra = \{0\}$, $\tilde{C}$ is canonically isomorphic, as M-space, to $\hat{C}$ and C. The Riesz subspace $\mathscr{I}\tilde{C} \cap \mathscr{D}\tilde{C}$ of C''/Ra is Dedekind complete, so can be identified with the Dedekind completion of C.

I have not mentioned the multiplicative structure of C. But this is implicit in the Riesz space structure; every M-space with unit has a canonical multiplicative structure, and uniferent Riesz homomorphisms between such spaces are multiplicative. Thus there are multiplications on C'' and C''/Ra which are consistent with the natural multiplications on C and U.

From what I have written it should be clear that the structure (C, C'') is a happy hunting ground for anyone who enjoys multifaceted phenomena. I should like to conclude by remarking on three of the lines of enquiry suggested by this book. (a) Is there any sense in which we can say that U is the largest subspace of $\ell^\infty(X)$ which can be naturally identified with a subspace of $C''(X)$? It may be necessary to use concepts from mathematical logic to explain what "naturally identify" can properly mean. (b) The space X can be retrieved, up to homeomorphism, from the Riesz space C, and the L-space C' can be found from C'', being identifiable as $(C'')^\times$. But widely varying spaces X can give rise to identical C' spaces. Maharam's theorem gives a simple complete classification of L-spaces in terms of densities of principal bands; is there an easy way to pick out the C' spaces from this classification, and to what extent can we derive topological properties of X from the properties of C'? (c) Because C'' is an M-space with unit, it can be identified with $C(Z)$ for an essentially unique compact Hausdorff space Z, and the embedding of C in C'' corresponds to a continuous surjection $q\colon Z \to X$. Is there a useful direct topological construction of (Z, q) from X? Which aspects of the structure (C, C'') can be effectively developed in terms of the triple (X, Z, q)?

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Methods of bifurcation theory, by Shui-Nee Chow and Jack K. Hale, A Series of Comprehensive Studies in Mathematics, vol. 251, Springer-Verlag, New York, 1982, xv + 515 pp., $48.00. ISBN 0-387-90664-9

The book under review is a major treatise on analytical methods in bifurcation theory. The theory is developed in the context of a large variety of examples and the authors also present a substantial amount of the requisite background material in nonlinear functional analysis and differential equations.

When a system of equations—algebraic, differential, functional, ... —depends on a parameter, it frequently happens that there are certain values of the parameter with the property that small variations in the parameter lead to significant changes in the qualitative behavior of the solutions of the corresponding system of equations. Loosely speaking, such values of the parameter are called bifurcation values and the general goal of bifurcation theory is to identify such points, to be as informative as possible about the nature of the solutions of the system of equations near such points and also to study the relations of such bifurcations with the local and global structure of the set of solutions. The importance of such a program stems from the ubiquity of examples of bifurcation occurring in systems of equations that arise naturally. Indeed a case can be made that a student's first systematic contact with nonlinear analysis should be bifurcation theory; that bifurcation is the central phenomenon of nonlinear behavior. It is safe to say that the number of interesting nonlinear equations for which one can find explicit solutions is exceedingly small. On the other hand advances in computational capability have significantly broadened our understanding of nonlinear systems of equations, revealing the astounding complexity of even the most innocent-looking systems. The goal is to obtain a coherent body of results which will lead to an understanding of the onset and progress of this complex behavior.

That solutions of parametrized equations change qualitatively as parameters are varied is not a recent observation, of course. Perhaps the bifurcation phenomenon that first leads to a systematic mathematical subject is the behavior of roots of polynomials. In the real domain, the roots of a quadratic coalesce and disappear as the coefficients are varied—a saddle-node bifurcation. In the complex domain, the roots bifurcate into a complex conjugate pair. The structure of the roots of systems of polynomial equations is the subject matter of algebraic geometry, and bifurcation is an important part of the subject. Moreover, algebraic-geometric ideas and tools are now used in other areas, under the name singularity theory. Singularity theory can be regarded as a generalization of the implicit function theorem, and the machinery is now an essential part of nonlinear analysis.

To get a glimpse of the general context let us consider some simple examples. First, the simplest possible. Consider a smooth function $f\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ with the property that $f(\lambda, 0) = 0$ for $\lambda \in \mathbf{R}$ and consider the fixed-point equation

$$x = f(\lambda, x), \qquad (\lambda, x) \in \mathbf{R} \times \mathbf{R}. \tag{1}$$

The set of *trivial* solutions is $T \equiv \mathbf{R} \times \{0\}$, and a point $(\lambda_0, 0) \in T$ is a *bifurcation point* of (1) if each neighborhood of $(\lambda_0, 0)$ contains solutions which are not in T. The implicit function theorem implies that if $(\lambda_0, 0)$ is a bifurcation point then $\partial f(\lambda_0, 0)/\partial x = 1$. However this is not a sufficient condition for bifurcation. But if, in addition, $\partial^2 f(\lambda_0, 0)/\partial\lambda\partial x \neq 0$, then the Morse lemma implies that there is a curve of solutions passing through $(\lambda_0, 0)$

and transverse to T. A solution $(\bar{\lambda}, \bar{x})$ of (1) is called an (asymptotically) *stable* fixed point of $f(\bar{\lambda}, \cdot)$ if the sequence of iterates under $f(\bar{\lambda}, \cdot)$ of points near $\bar{x}$ converge to $\bar{x}$. If $|\partial f(\bar{\lambda}, \bar{x})/\partial x| < 1$ then $(\bar{\lambda}, \bar{x})$ is stable while if $|\partial f(\bar{\lambda}, \bar{x})/\partial x| > 1$ it is unstable.

Since $\partial^2 f(\lambda_0, 0)/\partial\lambda\partial x \neq 0$, and $\partial f(\lambda_0, 0)/\partial x = 1$ the stability of the solutions on T changes at $\lambda = \lambda_0$. Suppose that $(\lambda, 0)$ is stable for $\lambda_0 - \varepsilon < \lambda < \lambda_0$ and unstable for $\lambda_0 < \lambda < \lambda_0 + \varepsilon$. If $\partial f(\lambda, x)/\partial x \neq 1$ on the bifurcating branches, then stability is determined by the sign of $x - f(x)$. The figure presents several typical basic bifurcation diagrams.

Near $(\lambda_0, 0)$ the solutions of (1) with $\lambda < \lambda_0$, $x \neq 0$ are called *subcritical*, and those with $\lambda > \lambda_0$, $x \neq 0$ are called *supercritical*. Thus subcritical solutions are unstable and supercritical solutions are stable. Moreover at $(\lambda_0, 0)$ there is an exchange of stability; in supercritical bifurcation the branch of bifurcating solutions inherits the stability. An elementary argument based on the mean-value theorem shows there is a connected set of nontrivial solutions of (1), whose intersection with T is empty, whose closure contains $(\lambda_0, 0)$, and which is either unbounded or contains another point of T in its closure. This is what it means that the local bifurcation at $(\lambda_0, 0)$ continues *globally*.

A solution of (1) is a fixed point of $f(\lambda, \cdot)$ and thus obviously a fixed point of $f^2(\lambda, \cdot)$, the second iterate of $f(\lambda, \cdot)$. Hence T is also a subset of the set of solutions of

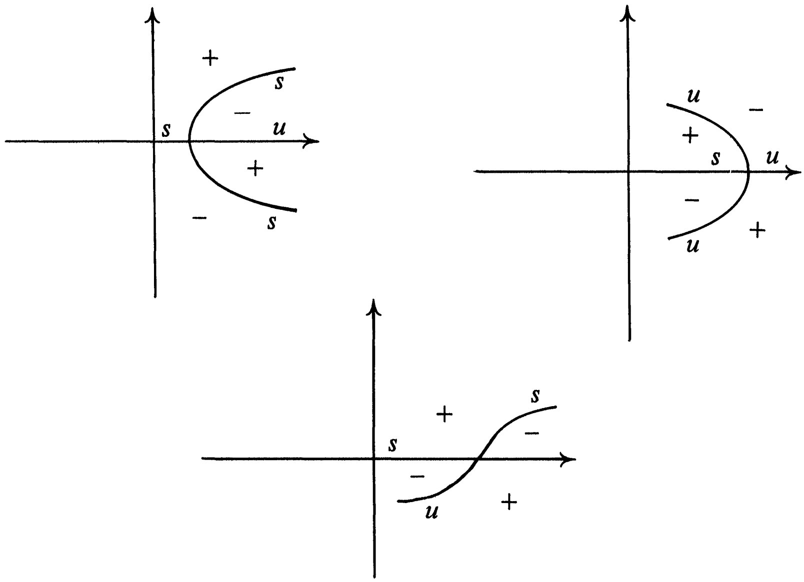
$$(2) \qquad x = f^2(\lambda, x), \qquad (\lambda, x) \in \mathbf{R}.$$

If $(\lambda^*, 0)$ is a point of bifurcation for (2) it is necessary that $|\partial f(\lambda^*, 0)/\partial x| = 1$. If $\partial f(\lambda^*, 0)/\partial x = -1$ and in addition $\partial^2 f(\lambda^*, 0)/\partial\lambda\partial x \neq 0$, then $(\lambda^*, 0)$ is a bifurcation point for equation (2) but not for (1). This is *period-doubling* bifurcation; a branch of fixed points of f^2 appears. Further bifurcations can occur. For instance, the branch of solutions of (2) emanating from $(\lambda^*, 0)$ can have branches of solutions of

$$(3) \qquad x = f^4(\lambda, x), \qquad (\lambda, x) \in \mathbf{R} \times \mathbf{R}$$

bifurcating from it.

The next simplest fixed-point problem is equation (1) where $(\lambda, x) \in \mathbf{R} \times \mathbf{R}^n$ and f: $\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ with $n > 1$. In $\mathbf{R}^n$, the derivative $\partial f/\partial x$ is an $n \times n$ matrix. Let $\sigma(A)$ denote the set of eigenvalues in the complex plane of an $(n \times n)$ matrix A. Later σ will denote the spectrum of an infinite-dimensional operator. A necessary condition for bifurcation is $1 \in \sigma(\partial f(\lambda_0, 0)/\partial x)$. When the dimension of the kernel of $(I - \partial f(\lambda_0, 0)/\partial x)$ is odd an additional transversality condition is sufficient for bifurcation. The definition of stability of a fixed point remains the same. However the linear criterion for stability is now a case of whether $\sigma(\partial f(\lambda_0, 0)/\partial x) \subset \{z \in \mathbf{C} \,|\, |z| < 1\}$ or $\sigma(\partial f(\lambda_0, 0)/\partial x) \cap \{z \in \mathbf{C} \,|\, |z| > 1\} \neq \varnothing$. Thus changes in stability occur when eigenvalues of the Jacobian pass through the unit circle in the complex plane. Eigenvalues can cross as complex conjugate pairs, and the menagerie of behavior is much wilder.



SIMPLE BIFURCATION DIAGRAMS. '+' denotes the region where $f(\lambda, x) > x$, '−' denotes the region where $f(\lambda, x) < x$, 's' denotes a stable solution and 'u' denotes an unstable solution. The first diagram exhibits supercritical bifurcation; the bifurcating branch carries the stability. The second diagram exhibits subcritical bifurcation, and the third exhibits transcritical bifurcation.

Equation (1) is a useful point from which to begin the study of more interesting systems. For a second example, consider a smooth system of ordinary differential equations depending on a single real parameter μ:

$$(4) \qquad \begin{cases} \dot{x}(t) = g(\mu, x(t)), \\ t \mapsto x(t) \text{ periodic,} \end{cases}$$

where g: $\mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$ is smooth. Suppose further that $g(\mu, 0) = 0$ for each $\mu \in \mathbf{R}$ so that $x(t) \equiv 0$ is a stationary solution of (4) for each $\mu \in \mathbf{R}$. It happens that there are particular values of μ_0 from which a branch of nonstationary solutions of (4) having period near a value T_0 emanates from $x(t) \equiv 0$. In order for such a bifurcation to take place it is necessary that $ni/T_0 \in \sigma(\partial g(\mu_0, 0)/\partial x)$ for some integer n. Then one gets a differentiable curve $\varepsilon(\mu)$: $I \to \mathbf{C}$, where I is an interval about μ_0, $\varepsilon(\mu_0) = ni/T_0$, and $\varepsilon(\mu)$ is an eigenvalue of $\partial g(\mu, 0)/\partial x$. The celebrated Hopf bifurcation theorem addresses precisely this situation. If $\mathrm{Re}(\varepsilon'(\mu_0)) \neq 0$ the most basic version of Hopf's theorem guarantees such bifurcation of (4). This phenomenon is now called Hopf bifurcation. A periodic solution of (4) is called *stable* if orbits of

(4) starting nearby evolve to this periodic solution as T evolves. The stationary solution $(\mu, 0)$ of (4) is stable if $\mathrm{Re}(\sigma(\partial g(\mu, 0)/\partial x)) \subset (-\infty, 0)$ and is unstable when $\mathrm{Re}(\sigma(\partial g(\mu, 0)/\partial x)) \cap (0, \infty) \neq \varnothing$. Thus the assumption that $\mathrm{Re}(\varepsilon'(\mu_0)) \neq 0$ guarantees a change in stability of $(\mu, 0)$ as μ goes through μ_0. Again there is an exchange of stability. Those periodic orbits bifurcating supercritically are stable and those bifurcating subcritically are unstable.

In his 1942 paper, Hopf defers to Poincaré, who Hopf felt probably knew the essence of periodic bifurcation. Poincaré certainly knew something of bifurcation, even if he didn't use the term. In addition to working with the phenomenon in the context of dynamics and differential equations, he drew an early bifurcation diagram of a variational problem in his investigations of rotating stars.

Again, similar bifurcation occurs for systems (4) for mappings g: $\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, $n > 2$; again there are more intricate possibilities. In particular, period doubling can occur. As the branch of periodic solutions evolves from the trivial solution, the period changes continuously. Frequently, it seems, there is a solution on the branch at which a secondary branch of solutions bifurcates; the periods on this branch vary continuously, except at the secondary bifurcation point, where the period doubles. Such period-doubling bifurcation is difficult to verify analytically, but is a ubiquitous phenomenon of numerical experiments. Moreover, it is easily explained in the context of equations (1) and (2). Let P be a small $(n - 1)$-dimensional transversal to some point on the orbit. Under the flow induced by (4), points on P near enough to the orbit return to P; this is the *Poincaré return map*. A fixed point of the Poincaré map corresponds to a periodic solution of (4). These solutions of (4) near the curve of periodic solutions which occur via a primary Hopf bifurcation can be expressed as solutions of an equation of type (1) where f: $\mathbf{R} \times \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ and the primary curve plays the role of the trivial solutions. Thus period doubling for the system of differential equations (4) corresponds to period doubling for the fixed-point equation (1). This second branch of solutions of (4) can, of course, itself period double, and indeed period-doubling cascades can occur. M. Feigenbaum's renormalization theory describes the asymptotic behavior of such cascades, both qualitatively and quantitatively.

One of the classic examples of bifurcation goes back to a problem of Euler, who tried to describe the possible configurations of a beam subject to a compressive load at the ends. The equation he derived was

$$(5) \qquad \begin{cases} \ddot{u}(x) = \lambda \sin(u(x)), & 0 \leqslant x \leqslant 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

Here u describes the displacement of the beam and λ is related to the load. The proper framework for (5) is an appropriate Banach space of functions u parametrized by λ. Again $u(x) \equiv 0$ is a solution for all λ, so $T = \{(\lambda, 0) | \lambda \in \mathbf{R}\}$ is a set of trivial solutions. If we formally linearize in the Banach space about $u \equiv 0$, we obtain the system

$$(6) \qquad \begin{cases} \ddot{u}(x) = \lambda_0 u(x), & 0 \leqslant x \leqslant 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

The mean-value theorem implies that for bifurcation to occur the linearized equation (6) must have nontrivial solutions, which means that $\lambda_0 = (n\pi)^2$ for some positive integer n. It turns out in this case that each λ_0 of this form is a bifurcation value and the bifurcation branches are global. In more general systems, of which (5) is a prototype, for bifurcation to occur it is necessary that the linearization of the system about the trivial solution be singular. However, singularity of the linearization is not in general sufficient. Sufficient conditions for bifurcation involve dimension restrictions on the null space of the linearization together with transversality conditions.

Let us consider bifurcation from the point of view of understanding the stability of systems, say of ordinary differential equations. Much of this approach can be traced to some of the early work in control theory in the mid-1800s by G. Airy and others, who were interested in understanding the behavior of mechanical devices, such as a steam engine. As a control on a machine is varied, the state of the machine can become unstable and the machine's behavior changes abruptly. The point where the state becomes unstable is a bifurcation point. If the bifurcation is supercritical, the machine settles on a nearby stable bifurcating solution. If it is subcritical, there is no nearby stable solution and the machine can exhibit catastrophic behavior. On the other hand, the machine is more sensitive to its control the nearer the system is to the bifurcation point. Clearly there was a major motivation for developing methods for understanding stability and bifurcation, and considerable work was done beginning in the late 1800s. J. C. Maxwell wrote a paper on the stabilizing effects of governors, and the Russian school of analysts, beginning with I. A. Vyshnegradskii, investigated the control of mechanical systems and wrote down what we can recognize as bifurcation conditions. Today bifurcation and stability are of great importance to people who design control systems, and bifurcation theory is taught in engineering courses. As an example of current interest, we mention the analysis of power grids, which are systems of electrical generators and transmission lines. The mathematical model is a system of nonlinear ordinary differential equations, called the swing equations. If a fault occurs, the system is changed and the problem is: how to drive it to a new stable state (in a matter of less than a second, else circuit breakers trip). To date, analysis has mostly involved Lyapunov functions. The use of bifurcation theory on the problem is in its infancy. Moreover the bifurcation problems are interesting; subcritical bifurcation seems to be the rule.

For an engineer, bifurcation is important because it is related to loss of stability; indeed to many engineers, the two terms are virtually synonymous.

Stability is also a useful concept for variational problems. Consider the differential equation (4) for the Euler-Bernoulli rod. The nonlinear functional $E\colon C^1[0,1] \to \mathbf{R}$ defined by

$$E(u) = \int_0^1 \left[\frac{1}{2}(\dot{u}(t))^2 - \lambda \cos(u(t))\right] dt, \tag{7}$$

has as its critical points the solutions of (4); equation (4) is the Euler-Lagrange equation of (7). Many problems have such a variational structure involving a

Banach space of functions X and a C^1 functional E: $\mathbf{R} \times X \to \mathbf{R}$. The equation of interest is

$$(8) \qquad \frac{\partial E}{\partial x}(\lambda, x) = 0.$$

In this setting solutions of (8) which arise as local minimizers of the functional $x \mapsto E(\lambda, x)$ are often of particular interest. In terms of stability, minimizers have the following significance: if solutions of (8) are viewed as the steady-state solutions of

$$(9) \qquad \dot{x}(t) = -\frac{\partial E}{\partial x}(\lambda, x(t)),$$

then if x_0 is a local isolated minimizer of $x \mapsto E(\lambda, x)$, those trajectories of (9) which start near x_0 and which are defined for all time $t > 0$ converge to x_0. A change in stability for solutions of (8) corresponds to a change in the character of the critical points of $x \mapsto E(\lambda, x)$. In the case of the Euler-Bernoulli rod, the solution $u \equiv 0$ is a minimizer of E for $\lambda < 0$ while for $\lambda > 0$ it is not. Thus the trivial solution is already unstable as a critical point of (7) at the first bifurcation point $\lambda = \pi^2$. This is because (5) or (7) is not a complete model. The end of the rod is free to rotate. The problem should be constrained by $\int \sin(u(t))\, dt = 0$. John Maddocks has shown the bifurcation at $\lambda = \pi^2$ of the constrained problem does accompany a change of stability. More occurs. For the constrained problem, there is a secondary bifurcation off of the first branch which carries the stability. It occurs when the two ends of the rod are brought together and the solutions on the secondary branch are rotations about the common point. Thus the global bifurcation diagram has a connection between the branch of solutions which has the end $x = 1$ above the end $x = 0$ and the branch of solutions with that end below.

The mathematical analysis of a particular bifurcation problem has both local and global features. It is frequently convenient to formulate a system of equations as a functional equation of the form

$$(10) \qquad F(\lambda, x) = 0,$$

where F: $\Lambda^n \times X \to Y$ for Banach spaces X and Y and coefficient field Λ. Suppose $F(\lambda_0, x_0) = 0$ and f is smooth with $\partial F(\lambda_0, x_0)/\partial x$ a Fredholm operator. Then the implicit function theorem implies that near (λ_0, x_0), equation (10) is equivalent to a finite-dimensional equation

$$(11) \qquad G(\lambda, x) = 0,$$

where G: $\Lambda^n \times \Lambda^k \to \Lambda^{k+l}$, with k the dimension of the kernel of $\partial F(\lambda_0, x_0)/\partial x$, and l the Fredholm index of $\partial F(\lambda_0, x_0)/\partial x$. This procedure is often called the Lyapunov-Schmidt procedure or the alternative method. In the case $k - l = 1$, so that (11) is a single scalar equation in $n + k$ unknowns, powerful analytic tools are available. When $\Lambda = \mathbf{C}$ and F is holomorphic, G is holomorphic and the Weierstrass Preparation Theorem gives a precise description of the solutions of (11) as the zeroes of polynomials. When $\Lambda = \mathbf{R}$ and F is smooth, the Malgrange Preparation Theorem delivers similar information about the solutions of (11). The basic idea is of course that information about the initial Taylor coefficients of G should yield information about the solutions of (11), and hence (10), near (λ_0, x_0).

The preparation theorems can be considered generalizations of the implicit function theorem. If F: $\mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^n$ is analytic, $f(0,0) = 0$ and $D_1 f(0,0) \neq 0$, the implicit function theorem implies $f(w, z) = 0$ can be solved for $w = h(z)$ analytically in a neighborhood of $(0,0)$. The Weierstrass Preparation Theorem considers the case that $f(w,0) = w^k g(w)$ for $g(w) \neq 0$ and implies that w can be solved for as the roots of a kth-degree polynomial equation

$$w^k + a_{k-1}(z) w^{k-1} + \cdots + a_0(z) = 0 \tag{12}$$

(for the case just above set $h = -a_0$). Thus the analytic question of the structure of the solutions of $f(w, z) = 0$ is reduced to an algebraic-geometric question. Bifurcation theory involves understanding the zeroes of a function f at such a branch point. The procedure outlined above is the essence of *singularity theory*. The bifurcation analysis of (10) is reduced to the finite-dimensional analysis of (11), called the *bifurcation function*. The resulting algebraic function (12) is called an *unfolding* of (11).

For the global analysis of (10), topological ideas play an essential role. When $n = 1$ and each $F(\lambda, \cdot)$ lies in a class of operators for which a topological degree is defined (for example, when $F(\lambda, \cdot)$ is a compact perturbation of the identity, the Leray-Schauder degree is defined), then a change in the degree of $F(\lambda, \cdot)$ on a neighborhood of $x = 0$ as λ passes through λ_0 implies that a bifurcation branch emanates from $(\lambda_0, 0)$. If there were no branching, by the homotopy property the degree would not change. This is the argument of Krasnoselskii and it ties together the two aspects of degree—its topological character and its relation to the local structure of a solution. Rabinowitz used the topology to extend Krasnoselskii's argument to produce from the same local assumptions a global conclusion in the sense we did for equation (1). Since then, more exotic topological invariants have been used in the same way to prove global versions of the Hopf theorem and other bifurcation results. If global results are combined with other features of a problem, more complete results can be obtained. For example, since (5) is a nonlinear Sturm-Liouville problem, the nodal structure the solutions inherit from the linear problem (6) is preserved along branches. Thus there are distinct global branches characterized by their nodal structures. Global results can be used to prove existence. For example, if it is known there is only one bifurcation point and solutions are a priori bounded, then the branch must extend over all parameter values in one or the other direction from the bifurcation point.

In the case that equation (11) is variational, in the sense that $Y = X^*$, the dual space of X, and there is a function ϕ: $\mathbf{R}^n \times X \to \mathbf{R}$ so that $\partial\phi(\lambda, x)/\partial x = f(\lambda, x)$,Morse theory and Ljusternik-Schnirelman category are available for both local and global results. Incidentally we have implicitly assumed bifurcation occurs as a connected set, at least locally. When bifurcation is established with degree or singularity theory, such is the case. However, there is an example of R. Böhme, wherein the bifurcation is established by a variational argument, and the solutions form an infinite set of disjoint circles collapsing down to the bifurcation point. The example is smooth, but not analytic. In the analytic category, such behavior seems not to happen.

Finally there is bifurcation from the point of view of dynamics, with cascades of period-doubling, strange attractors, homoclinic orbits, etc. This has been, and continues to be, an area of rapid development, and the machinery, results, and even general concepts, are still evolving.

The point of all this is that bifurcation is a virtually universal phenomenon of nonlinear behavior and "bifurcation theory" has different flavors in different areas. A reader might be attracted to one book and find another completely impenetrable. Within the last decade or so, bifurcation theory has matured to the point that a large number and variety of expositions have been written, as a perusal of reviews in this *Bulletin*, the *SIAM Review* and the *Bulletin of the London Mathematical Society* will show. Let us now turn to the book of Chow and Hale.

Since bifurcation theory is meaningful insofar as it gives insight into the behavior of actual equations, any exposition of bifurcation theory is motivated by a number of explicit applications. Thus Chapter 1 contains discussion of the static solutions of the Euler-Bernoulli rod, Hopf bifurcation, and planar homoclinic orbits. Chapter 2 is a 70 page review of fundamental techniques of nonlinear analysis. Chapter 3 is devoted to some applications of the implicit function theorem. Chapter 4 is a similar review of variational methods, with applications to Hamiltonian systems and certain partial differential equations. In Chapter 5 the authors get down to bifurcation theory per se, by developing the relations between the nonlinear equation and its linearization about a trivial solution. The multiplicity of eigenvalues and multiple parameters are discussed and degree theory is used to prove Paul Rabinowitz's global result. In Chapter 6, the effects of higher-order terms are investigated. There is a brief introduction to the effects of symmetry. The book does not attempt to develop systematically the (currently active) subject of bifurcation under symmetry. Some later examples involve symmetry, but for instance in the later example of the forced Duffing's equation, the bifurcation analysis is done directly. What is emphasized is that symmetry complicates the analysis. Beginning in this chapter, the implicit function theorem is inadequate and more general singularity theory is used. Chapter 7 discusses bifurcation from more complicated singularities. Chapter 8 contains several applications of the earlier theory: von Karman's equation, Brusselator, Duffing's equation.

To this point in the book, the theory developed is concerned with the bifurcation of the zero set of a functional—what the authors call static bifurcation, since no dynamics is involved. Chapter 10 begins what they call dynamic bifurcation theory; the behavior depends on the time evolution of the system. In particular, in Chapter 10 they introduce the center-manifold theorem and use it to prove the "generic Hopf bifurcation theorem" and ancillary results. Chapters 10 and 11 consider planar systems, for autonomous and forced systems, respectively. A major theme here is the behavior of homoclinic orbits. There is a brief introduction to symbolic dynamics and also a brief introduction to turbulence (chaos?), but these topics in dynamics are not pursued further. Horseshoes are drawn but not discussed much. Chapter 12 introduces normal forms and uses them to prove the standard results in averaging theory and to discuss integral manifolds and bifurcation to tori. Chapter 13 discusses degenerate bifurcation (using unfoldings) and Chapter 14 discusses the perturbation theory of the spectrum of linear operators.

The flavor and range of the book can be gauged from this survey of the contents. There is considerable material covered. Still, of course, not everything can be covered. It is not a flaw that some topics (such as bifurcation under symmetry and topics in dynamics) are omitted or cursorily discussed; the authors had to make choices. Indeed, as Chow and Hale certainly knew, there are other recent books on these two particular topics, and they chose not to duplicate material. The organization is via method, not via type of application. The flavor of the book is analytic. It is written by analysts for analysts. The authors state explicitly "an alternative title for this book would perhaps be *Nonlinear Analysis, Bifurcation Theory and Differential Equations*. Our primary objective is to discuss those aspect of bifurcation theory which are particularly meaningful to differential equations" (p. vii). Thus a person with some background in analysis will feel most at home. For example, some results from analysis (especially functional analysis) are mentioned casually without reference or citation. Some instances, largely from Chapter 3, which is mostly introductory: maximum principle (p. 111—do the authors mean the minimax characterization of eigenvalues?), Mazur's Theorem (p. 121), Friedrich's inequality (p. 122), usual bootstrap argument (p. 153), inverse of Laplacian (p. 239). To be more precise about the flavor of the book, it approaches bifurcation theory from the point of view of singularity theory. Even with topics that are not part of singularity theory, the authors use the ideas of singularity theory. For example, when degree is developed, generic approximations are used. This approach, motivated to some extent by the development of numerical continuation methods, has its advantages. It emphasizes that the degree of an operator counts zeroes or fixed points, taking into account the local structure of the operator around the points. Once the machinery is developed, proofs of particular results are often conceptual and sometimes technically trivial.

We are impressed with the scope of the book and its novel treatment of several topics. However, the book is not unflawed. We get the impression the authors' hearts were in the applications of singularity theory to ordinary differential equations. In writing a broader treatment of bifurcation, the authors got careless at points, and incomplete at others. Some of the subtleties of working in Banach spaces are missing or elided. For example, Leray-Schauder degree is never developed, although it is used. A novice reader will not develop an understanding that some kind of compactness of operators or solutions is necessary for things to work. As for carelessness: consider some specific cases. The authors attempt (pp. 67–68) to prove the Borsuk-Ulam theorem by making an approximation $\tilde{f}$ to an odd function such that the zeroes are all nondegenerate. They then consider $f(x) = \tilde{f}(x) - \tilde{f}(-x)$ and claim it also has nondegenerate zeroes and compute the degree by counting. However it is by no means clear that f has nondegenerate zeroes. Approximation ideas work, but a more careful approximation is needed near $x = 0$. Just after this on p. 68, the authors slightly misstate a result. They state that an odd continuous map from S^{n-1} to $\mathbf{R}^n$ has a zero. As stated, the result is clearly false. However, they then offer an argument that would work for both the correct and incorrect statements, except that they misapply the Tietze extension theorem. As one more example, consider the argument of the Rabinowitz global bifurcation theorem. The proof strikes us as somewhat muddled, but in particular, there is a technical mistake in that a function ρ is introduced which does not do what it is supposed to do (there may be bifurcation points not connected to the K_0 of the theorem). The discussion of this theorem is one of the points where details of compactness get swept under the rug. (Also, it is not as complicated as the authors indicate to go from the finite-dimensional version (which the authors have just proved in a manner more in the spirit of the rest of the book) to the Banach-space version.)

The authors suggest several ways the book could be used in a course. With an expert teacher, the book could indeed make a good text. We cannot really recommend the book to a novice in analysis. The book is more likely to be used as a professional reference. Much material is here that is not conveniently available elsewhere. At the end of each chapter there are extensive bibliographic comments, which trace the history and development of the chapter's results. The list of references is impressive. The reviewers picked a few papers at random. Virtually all within the purview of the book were included in the list of references. Already the book is regularly cited in papers as an expository source. To make the book more valuable as a reference, we would suggest the authors expand their index in the second edition. The table of contents is more useful than the index. While the table of contents is 4 + pages long and the reference list is 19 pages of small print, the index is less than 3 pages.

The book, which was published in 1982, has become one of the standard references in the research literature on the subject. Since bifurcation theory is a field which is rapidly developing and also one which has many contact points with diverse areas of mathematics and applied science, it is no small task to present a treatment which is at once broad and coherent. It is a major accomplishment of Chow and Hale to have written this exposition.

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