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SOLUTION OF A PROBLEM RAISED BY RUBEL

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The following problem was raised by L. Rubel in the 1950s and appears in [**2**]; my interest in it was rekindled by a query that B. Ghusayni submitted to the Notices of the American Mathematical Society.

PROBLEM. Suppose $E \neq \{0\}$ is a linear subspace of $L^2(\mathbf{R})$ such that

(i) $f \in E \Rightarrow f\hat{\ } \in E$ (where $f\hat{\ }$ is the Fourier transform of f)

(ii) $g \in L^2(\mathbf{R})$, $|g| \leq |f|$ a.e. for some $f \in E$ implies that $g \in E$.

Then must $E = L^2(\mathbf{R})$?

We propose to prove more, namely:

THEOREM 1. *Let $g, f \in L^2(\mathbf{R})$, $f \neq 0$. Then there exist functions $\varphi_j \in L^\infty(\mathbf{R})$, $j = 1, \ldots, 5$ such that, denoting by M_j the operator of multiplication by φ_j and by F the Fourier transformation, we have*

$$g = M_5 F M_4 F M_3 F M_2 F M_1 \cdot f.$$

NOTATIONS.

$$\begin{aligned} l^{2^*} &= \{h; h \in L^2(\mathbf{R}), h \text{ constant in each } [n, n+1)\}, \\ L^{2^*} &= \{H; |H(x)| \leq h(x) \text{ for some } h \in l^{2^*}\}, \\ &= \{H; H = \varphi h,\ h \in l^{2^*},\ \varphi \in L^\infty(\mathbf{R})\}, \\ &= \{H; \Sigma \sup_{n \leq x < n+1} |H(x)|^2 = |||H|||^2 < \infty\}. \end{aligned}$$

LEMMA 1. *If $\psi \in L^2(\mathbf{R})$ and* support$(\psi) \subset [0,1]$, *then $\hat{\psi} = F\psi \in L^{2^*}$ and $|||\hat{\psi}||| \leq 2\|\psi\|$.*

LEMMA 2. *If $\Psi \in L^2(\mathbf{R})$, then there exists a continuous Φ, $|\Phi(x)| = 1$, such that $(\Phi\Psi)\hat{\ } \in L^{2^*}$ and $|||\Phi\Psi\hat{\ }||| \leq 2\|\Psi\|$.*

PROOF. Write $\Psi = \Sigma\psi_j$ with $\psi_j = \Psi$ on I_j, where $\{I_j\}$ are intervals of length 1 whose disjoint union covers $\mathbf{R}$. Write $\Phi(x) = \exp\{i\lambda_j x\}$ on I_j,

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$\lambda_j \in 2\pi\mathbf{Z}$, and the λ_j's increase fast enough to make $\hat{\psi}_j(\xi - \lambda_j)$ virtually orthogonal. Use Lemma 1 and the equality $\|\Psi\|^2 = \Sigma\|\psi_j\|^2$.

PROOF OF THE THEOREM. Given $f \neq 0$, we take φ_1 bounded and of (well-placed) small support so that $\varphi_1 f$ is an approximate point mass and its Fourier transform is bounded away from zero on $[0,1]$. Thus, the indicator function of $[0,1]$, denoted $1_{[0,1]}$, has the form M_2FM_1f. We now apply F, multiply the outcome by a 2π-periodic function φ_3 and apply F again. What we obtain is the function

$$F(x) = \hat{\varphi}_3(n) \quad \text{for } n \leq x < n+1,$$

which belongs to l^{2^*}. Our limitation is that φ_3 must be bounded, but we invoke the result of [**1**] which says that given any sequence $\{a_n\} \in l^2$ there exists continuous 2π-periodic φ_3 such that $|a_n| \leq |\hat{\varphi}_3(n)|$. It follows that the functions $FM_3FM_2FM_1f$ majorize every function in l^{2^*} and hence in L^{2^*} and the functions $M_4FM_3FM_2FM_1f$ cover L^{2^*}. Lemma 2 shows how an additional F and division by Φ (multiplication by $\overline{\Phi}$) covers all of $L^2(\mathbf{R})$.

REMARK. The method of [**1**] applies, as is, to show that if G is a compact abelian group and $f \in L^2(G)$, there exists a bounded φ on G such that $|\hat{\varphi}| \geq |\hat{f}|$ on $\hat{G}$. This permits an extension of Theorem 1 to all locally compact abelian groups (notice that the Fourier operator F appears four times so that we end on the group we have started with).

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