On the other side of the coin, this book is still, strictly speaking, not a text. Exercises for the student to worry about are virtually nonexistent, although many proofs leave sufficient gaps to provide challenge. The removal of some peripheral material to exercises would give the book more of a Clifford-Preston flavor, while allowing some contact with every chapter in a one-semester course. Complaints about choice of content should be forestalled until the appearance of a second volume, due out in the near future and promising cohomology, semilattices, Lie semigroups, and other topics of current interest. It strikes the reviewer that cohomology, which has provided the subject with some of its most elegant theorems, would have been well invested in the first volume. Nevertheless, the reviewer believes Wallace would be happy with this book, and in this subject there can be no better compliment.

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D. R. Brown

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Analytic functional calculus and spectral decompositions, by Florian-Horia Vasilescu, Mathematics and its Applications, Volume 1, D. Reidel Publishing Company, Dordrecht, Holland, 1982, xiv + 378 pp., Dfl. 180,–, U.S. $78.50. ISBN 90-277-1376-6

A linear transformation T acting on a finite-dimensional complex vector space $\mathscr{X}$ can always be decomposed as $T = D + N$, where (i) D is diagonalizable and N is nilpotent; and (ii) $DN = ND$; moreover, such a decomposition is unique with respect to the conditions (i) and (ii), and both D and N are indeed polynomials in T. When $\mathscr{X}$ is an infinite-dimensional Banach space, such a representation for a bounded operator T is no longer true, but an important class of transformations introduced and studied by N. Dunford [3] in the 1950s possesses a similar property.

By definition, a *spectral operator* T acting on $\mathscr{X}$ is one for which there exists a spectral measure E (i.e., a homomorphism from the Boolean algebra of Borel subsets of the complex plane $\mathbf{C}$ into the Boolean algebra of projection operators on $\mathscr{X}$ such that E is bounded and $E(\mathbf{C}) = I$) satisfying the following two properties: (1) $TE(B) = E(B)T$; and (2) $\sigma(T|_{E(B)\mathscr{X}}) \subset \overline{B}$, for all B(Borel) $\subset \mathbf{C}$. Such an E is called a *resolution of the identity* for T, and is unique with respect to T. Conversely, given a spectral measure E supported on a compact subset of $\mathbf{C}$, the Riemann-Stieltjes integral $\int \lambda dE(\lambda)$ defines a bounded operator S on $\mathscr{X}$, which turns out to be spectral with resolution of the identity E. Operators admitting such an integral representation are called *scalar*.

A basic result of the theory of spectral operators states that if T is spectral then $T = S + Q$, where (i′) S is scalar and Q is *quasinilpotent* ($\sigma(Q)$, the spectrum of Q, equals $\{0\}$); and (ii′) $SQ = QS$. Such a decomposition, called canonical, is unique; S and Q are called the scalar and radical parts of T, respectively. Also, since every bounded operator that commutes with T does so with E, both S and Q belong to the double commutant of T; T and S have identical resolutions of the identity and equal spectra. Moreover, the inverse-closed Banach algebra generated by T and the range of E splits as a direct sum of its radical and the inverse-closed Banach algebra generated by the range of E (which is isomorphic to the algebra of E-essentially bounded Borel functions on $\sigma(T)$). Therefore, a spectral operator admits a rich functional calculus. On Hilbert space, an operator is scalar if and only if it is similar to a normal operator (J. Wermer), which implies that the sum and product of two commuting spectral operators is spectral (a fact not necessarily true in arbitrary Banach spaces (S. Kakutani)).

There are basically two ways in which the notion of a spectral operator can be generalized: (a) by extending the notion of a resolution of the identity; and (b) by restricting the algebras used to define the functional calculus for the scalar part. Colojoară and Foiaş [2] considered at length the various types of operators that can thus be obtained (decomposable, generalized scalar, generalized spectral, $\mathscr{U}$-scalar, $\mathscr{U}$-spectral, $\mathscr{U}$-decomposable) and extended greatly the pioneering work of Dunford, J. Schwartz, W. Bade, S. Kakutani and others. As a result of that study the theory became a major focus of attention, and in the ensuing decade and a half a number of simplifications, generalizations, solutions to open problems and applications were obtained. In particular, a considerable portion of the several variables theory was developed. Since many of the interesting classes consist of unbounded operators (for instance, boundary value problems often give rise to spectral unbounded operators) acting on Fréchet spaces, the extensions had to cover those cases, too. Before we look at those developments in more detail we must pause to discuss the elements of spectral theory for several commuting operators.

The first notion of joint spectrum appears in the work of R. Arens and A. Calderón on analytic functions of several Banach algebra elements. For $a_1, \ldots, a_n$ in a commutative Banach algebra $\mathscr{A}$, *the joint spectrum* of $a = (a_1, \ldots, a_n)$ relative to $\mathscr{A}$ is simply $\sigma_{\mathscr{A}}(a) = \{\varphi(a) := (\varphi(a_1), \ldots, \varphi(a_n)) \colon \varphi \in M_{\mathscr{A}}\}$, i.e., $\sigma_{\mathscr{A}}(a)$ is the range of the (joint) Gelfand transform of a. Such a spectrum possesses a well-behaved functional calculus and is adequate in a number of instances. Generally speaking, however, it is much too large. For example, if $a_1, \ldots, a_n$ generate $\mathscr{A}$ then $\sigma_{\mathscr{A}}(a)$ is polynomially convex, so that the functional calculus is basically a polynomial calculus in this case. When a is a commuting system in a noncommutative algebra (say an n-tuple of commuting operators on $\mathscr{X}$, regarded as elements of $\mathscr{L}(\mathscr{X})$, the algebra of all bounded operators on $\mathscr{X}$), various kinds of joint spectra have been considered by many authors, including the left spectrum, the right spectrum, and the algebraic spectra relative to (commutative or not) Banach algebras containing the a_i's. (Briefly, if the a_i's are in the center of the Banach algebra $\mathscr{A}$, then a is said to be invertible with respect to $\mathscr{A}$ if the equation $a_1 b_1 + \cdots + a_n b_n = 1$ admits a solution in $\mathscr{A}$ $(b_1, \ldots, b_n \in \mathscr{A})$; equivalently, a is invertible relative to $\mathscr{A}$ if and only if the closed ideal generated by $\mathscr{A}$ contains the identity.) Each of these notions has its own advantages, but they all share two major drawbacks: lack of a sufficiently rich functional calculus and inadequacy to properly reflect the joint action of the operators on the space. In 1970 J. L. Taylor found a successful notion which gradually began to play a major role and became an essential tool in the subject.

A commuting n-tuple $T = (T_1, \ldots, T_n)$ of operators on a Banach space $\mathscr{X}$ can be associated with a chain complex $K(T, \mathscr{X})$, called the *Koszul complex* for T on $\mathscr{X}$, as follows. If $\Lambda[e]$ denotes the exterior algebra in n generators $e_1, \ldots, e_n$, the map d^T: $\Lambda[e] \otimes \mathscr{X} \to \Lambda[e] \otimes \mathscr{X}$,

$$d^T(\xi \otimes x) = \sum_{i=1}^{n} (e_i \wedge \xi) \otimes T_i x, \qquad \xi \in \Lambda[e], \quad x \in \mathscr{X},$$

gives rise to the boundary maps $d_k^T = d^T|_{\Lambda^k[e] \otimes \mathscr{X}}$ (the restriction of d^T to k-forms); $K(T, \mathscr{X})$ is then the complex $\{\Lambda^k[e] \otimes \mathscr{X}, d_k^T\}_{k=0}^n$. T is said to be *nonsingular* on $\mathscr{X}$ if $K(T, \mathscr{X})$ is exact at every stage, and the *Taylor spectrum* of T, $\sigma(T, \mathscr{X})$, is then defined as the complement of the set of complex n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$ for which $T - \lambda$ is nonsingular. As an example, consider the case $n = 2$. Then

$$K(T, \mathscr{X}): 0 \to \mathscr{X} \xrightarrow{d_0} \mathscr{X} \oplus \mathscr{X} \xrightarrow{d_1} \mathscr{X} \to 0,$$

where $d_0(x) = T_1 x \oplus T_2 x$ and $d_1(x, y) = -T_2 x + T_1 y$. Then T is nonsingular if and only if $\ker T_1 \cap \ker T_2 = \{0\}$, $\operatorname{ran} T_1 + \operatorname{ran} T_2 = \mathscr{X}$, and whenever $T_1 y = T_2 x$ there exists a vector $z \in \mathscr{X}$ such that $T_1 z = x$ and $T_2 z = y$.

Taylor proved that this joint spectrum is a compact, nonempty subset of the cartesian product of the individual spectra, that it is contained in any algebraic joint spectrum, and that it possesses a rich functional calculus: if U is an open neighborhood of $\sigma(T, \mathscr{X})$ and $A(U)$ denotes the algebra of functions analytic in U, then there exists a continuous unital algebra homomorphism from $A(U)$ into $\mathscr{L}(\mathscr{X})$ such that the image of the coordinate function z_i is the operator T_i $(i = 1, \ldots, n)$. Moreover, the values of this homomorphism belong to $(T)''$, the double commutant of T. Taylor also proved the spectral mapping theorem in this context: If $f = (f_1, \ldots, f_m) \in A(U)^{(m)}$ then for $f(T) := (f_1'(T), \ldots, f_m(T))$ the equality $\sigma(f(T), \mathscr{X}) = f(\sigma(T, \mathscr{X}))$ holds. As an immediate consequence of this, it follows that whenever $\sigma(T, \mathscr{X})$ is the disjoint union of two nonempty compact subsets σ_1 and σ_2 then $\mathscr{X}$ can be written as the direct sum of two closed subspaces $\mathscr{X}_1$ and $\mathscr{X}_2$ such that each is invariant under T and $\sigma(T|_{\mathscr{X}_j}, \mathscr{X}_j) = \sigma_j$ $(j = 1, 2)$.

This last property of the Taylor spectrum is of considerable importance in the study of decomposable n-tuples of operators. Other properties include the upper semicontinuity of separate parts, the superposition property and the uniqueness of the functional calculus. In addition, if T and S are two commuting n-tuples of operators on $\mathscr{X}$ and A is a map intertwining T and S ($AT = SA$), then for any $f = (f_1, \ldots, f_m) \in A(U)^{(m)}$, $U \supset \sigma(T, \mathscr{X}) \cup \sigma(S, \mathscr{X})$, one has $Af(T) = f(S)A$; in particular, if T and S are (jointly) similar, so are $f(T)$ and $f(S)$. The Arens-Calderón-Waelbroeck-Bourbaki functional calculus for n-tuples in a commutative Banach algebra $\mathscr{A}$ can be derived from the above by observing that for $a = (a_1, \ldots, a_n) \in \mathscr{A}^{(n)}$, $\sigma_{\mathscr{A}}(a) = \sigma(L_a, \mathscr{A})$, where $L_a = (L_{a_1}, \ldots, L_{a_n})$ denotes the n-tuple of left multiplications by the a_i's acting on $\mathscr{A}$ (regarded as a Banach space). R. Harte's polynomial mapping theorem for the left and right spectra (or M.-D. Choi's and C. Davis's version for the approximate point spectrum), however, still requires an independent argument. As the reader will have probably noticed, both the left, σ_l, and right, σ_r, spectra are subsets of σ; also, if $\mathscr{X}$ is finite-dimensional, $\sigma_l = \sigma_r = \sigma = \sigma_{\mathscr{A}}$, where $\mathscr{A}$ is the Banach subalgebra of $\mathscr{L}(\mathscr{X})$ generated by the n-tuple.

The left and right spectra are often adequate in a number of problems (e.g., the spectral theory of generalized derivations), but the existence of such a well-behaved functional calculus is what makes the Taylor spectrum the overwhelming choice in the theory of spectral decompositions. Since the notion of nonsingularity has been given in terms of a chain complex, perhaps the reader wonders how the actual construction of the functional calculus is implemented, and therefore we shall briefly indicate here the main steps. For simplicity, assume that $\mathscr{X}$ is actually a Hilbert space. In this case the nonsingularity of a commuting n-tuple T can be decided by a single operator acting on (the Hilbert space) $\Lambda[e] \otimes \mathscr{X}$, namely, T is nonsingular on $\mathscr{X}$ if and only if $R(T) = d^T + (d^T)^*$ is invertible. The map $\lambda \mapsto R(\lambda - T)$ from $\mathbf{C}^n \setminus \sigma(T, \mathscr{X})$ to $\mathscr{L}(\Lambda[e] \otimes \mathscr{X})$ is real-analytic with values in the invertible operators on $\Lambda[e] \otimes \mathscr{X}$. For $\lambda \neq 0$, $R(\lambda)^2 = |\lambda|^2$, so that $R(\lambda)^{-1} = |\lambda|^{-2} R(\lambda)$.

The other basic ingredient needed to describe the functional calculus is the *Bochner-Martinelli formula*, which states that an analytic function f in $A(U)$ can be represented as

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int_\Sigma f(\lambda) \sum_{j=1}^n (-1)^{j-1} \frac{\bar{\lambda}_j - \bar{z}_j}{|\lambda - z|^{2n}} \bigwedge_{\substack{1 \leqslant k \leqslant n \\ k \neq j}} d\bar{\lambda}_k \wedge d\lambda,$$

where z is an arbitrary point in a compact subset K of U and $\Sigma = \partial K$ is piecewise smooth. Unlike Cauchy's kernel when $n = 1$, the Martinelli kernel

$$M(\lambda - z) = (n-1)! \sum_{j=1}^n (-1)^{j-1} \frac{\bar{\lambda}_j - \bar{z}_j}{|\lambda - z|^{2n}} \bigwedge_{\substack{1 \leqslant k \leqslant n \\ k \neq j}} d\bar{\lambda}_k$$

does not "create" analytic functions, but it nevertheless reproduces them. It is not hard to check that

$$M(\lambda - z) = R(\lambda - z)^{-1} \left(\bar{\partial}_\lambda R(\lambda - z)^{-1}\right)^{n-1} S\Big|_{\Lambda^0[e]},$$

where $S(\xi) = e_1 \wedge \cdots \wedge e_n \wedge \xi$, $\xi \in \Lambda[e]$. If we then let

$$M(\lambda - T) = R(\lambda - T)^{-1} \left(\bar{\partial}_\lambda R(\lambda - T)^{-1}\right)^{n-1} S\Big|_{\Lambda^0[e] \otimes \mathscr{X}},$$

we can define

$$f(T):=\frac{1}{(2\pi i)^n}\int_{\Sigma} f(\lambda)M(\lambda - T)\,d\lambda,$$

where $\Sigma = \partial K$ is piecewise smooth and K is a compact subset of U containing $\sigma(T, \mathscr{X})$ in its interior.

This relatively simple form for the functional calculus in Hilbert spaces, due to the author, was an important breakthrough in the whole theory, since the original version of Taylor involved the use of a Cauchy-Weil integral formula in homology, which was a highly noncomputational tool. There are now in any case several simplifications of Taylor's techniques which make the representation of $f(T)$ more amenable to the untrained eye, even in the case when $\mathscr{X}$ is a Fréchet space (M. Putinar, the author and others have extended most of the results on Taylor's spectrum to that situation).

Turning now our attention to the theory of spectral decompositions, recall that an operator T acting on a Banach space $\mathscr{X}$ is called *m-decomposable* if for every finite open covering $\{U_1, \ldots, U_m\}$ of $\sigma(T, \mathscr{X})$ there exist spectral maximal spaces $\mathscr{Y}_1, \ldots, \mathscr{Y}_m$ for T such that (i) $\sigma(T, \mathscr{Y}_j) \subset U_j$ $(j = 1, \ldots, m)$ and (ii) $\mathscr{X} = \sum_{j=1}^m \mathscr{Y}_j$. (A subspace $\mathscr{Y}$ of $\mathscr{X}$ is spectral maximal for T if $\mathscr{Y}$ is invariant under T and whenever $\mathscr{Z}$ is a subspace of $\mathscr{X}$ invariant under T and satisfying $\sigma(T, \mathscr{Z}) \subset \sigma(T, \mathscr{Y})$, $\mathscr{Z}$ is contained in $\mathscr{Y}$; a typical example is given by the spectral subspace associated to a component of the spectrum of T). T is called *decomposable* if it is m-decomposable for every $m \geqslant 1$, *strongly decomposable* if $T|_{\mathscr{Y}}$ is decomposable for every spectral maximal subspace $\mathscr{Y}$, and *weakly decomposable* if for every finite open covering $\{U_1, \ldots, U_m\}$ of $\sigma(T, \mathscr{X})$ there are spectral maximal spaces $\mathscr{Y}_1, \ldots, \mathscr{Y}_m$ satisfying (i) above and (ii') $\mathscr{X} = (\sum_{j=1}^m \mathscr{Y}_j)^-$.

It is clear that strong decomposability implies decomposability (the converse is false (E. Albrecht)), that decomposability implies 2-decomposability (the converse also holds (M. Radjabalipour)) and that decomposability implies weak decomposability (converse false (Albrecht)). The connection of decomposability to spectrality is achieved by considering the notion of a *spectral capacity*, introduced by C. Apostol in 1968. A map X: $\mathrm{Cl}(\mathbf{C}^n) \to \mathscr{S}(\mathscr{X})$ (from the collection of all closed subsets of $\mathbf{C}^n$ to the family of all closed subspaces of $\mathscr{X}$) is called an *m-spectral capacity* if it satisfies the following three conditions: (i) $X(\varnothing) = \{0\}$ and $X(\mathbf{C}^n) = \mathscr{X}$; (ii) $X(\bigcap_{k=1}^\infty F_k) = \bigcap_{k=1}^\infty X(F_k)$ for all sequences $\{F_k\}_{k=1}^\infty \subset \mathrm{Cl}(\mathbf{C}^n)$; and (iii) for every open covering $\{U_1, \ldots, U_m\}$ of $\mathbf{C}^n$, $\mathscr{X} = \sum_{j=1}^m X(\overline{U}_j)$. If condition (iii) is satisfied for every $m \geqslant 1$, X is called a spectral capacity; X is a strong spectral capacity if X satisfies (i), (ii), and (iii'): for every $F \in \mathrm{Cl}(\mathbf{C}^n)$ and every open covering $\{U_1, \ldots, U_m\}$ of $\mathbf{C}^n$, $X(F) = \sum_{j=1}^m X(F \cap \overline{U}_j)$.

For $n = 1$, C. Foiaş showed that T is 2-decomposable (resp. decomposable, strongly decomposable) if and only if there exists a 2-spectral (resp. spectral, strong spectral) capacity X such that $X(F)$ is invariant for T and $\sigma(T, X(F)) \subset F$ for all $F \in \mathrm{Cl}(\mathbf{C})$. Spectral capacities for a decomposable operator are always unique, but not every spectral capacity is the spectral capacity of some decomposable operator (Example (Vasilescu). On $\mathscr{X} = \{f \in C^1([0,1]): f(0) = f(1) = 0\}$ let $X(F) = \{f \in \mathscr{X}: \operatorname{supp}(f') \subset F\}$). It is clear that the above characterization can then be used as the definition of m-decomposability (resp. decomposability, strong decomposability) when $n > 1$ (where $\sigma(T, \mathscr{X})$ will now mean Taylor's joint spectrum of T of $\mathscr{X}$). S. Frunză, who introduced this notion for n-tuples, proved that if $T = (T_1, \ldots, T_n)$ is 2-decomposable then there is only one spectral capacity X_T such that $TX_T(F) = X_T(F)T$ and $\sigma(T, X_T(F)) \subset F$ for all $F \in \mathrm{Cl}(\mathbf{C}^n)$. Moreover, the spaces $X_T(F)$ are all spectral maximal for T. Concerning the relation of m-decomposability to decomposability, the following is a useful theorem: If T is m-decomposable and $m \geqslant \dim \sigma(T, \mathscr{X}) + 1$ then T is decomposable (Albrecht-Vasilescu). Thus, whenever $\dim \sigma(T, \mathscr{X}) = 0$, T is decomposable (trivial observation: any commuting n-tuple is 1-decomposable), so that all commuting n-tuples with *totally disconnected* spectrum are decomposable.

The multivariable theory shares two other properties with the one-variable theory. If T is decomposable then (1) $\sigma_\pi(T, \mathscr{X}) = \sigma(T, \mathscr{X})$ (where σ_π denotes approximate point spectrum); and (2) T has the *single-valued extension property* (s.v.e.p). By definition, T has the s.v.e.p. if for every $\lambda \in \mathbf{C}^n$ there exists an open disc D_λ ($\lambda \in D_\lambda$) such that the Koszul complex

$$K(T - \lambda, A(D_\lambda, \mathscr{X}))$$

is exact everywhere except perhaps at the last stage ($k = n$). For an arbitrary commuting T the (analytic) local spectrum $\sigma(x; T, \mathscr{X})$ is defined as the complement of the set of points $\lambda \in \mathbf{C}^n$ for which there is an open set U containing λ such that the equation $\sum_{i=1}^n (T_i - z_i) f_i(z) = x$ is solvable in $A(U, \mathscr{X})$. The local spectrum has the following properties:

(i) $\sigma(0; T, \mathscr{X}) = \varnothing$ and, if T has the s.v.e.p., $\sigma(x; T, \mathscr{X}) = \varnothing$ implies $x = 0$;

(ii) $\sigma(x + y; T, \mathscr{X}) \subset \sigma(x; T, \mathscr{X}) \cup \sigma(y; T, \mathscr{X})$;

(iii) $\sigma(\lambda x; T, \mathscr{X}) = \sigma(x; T, \mathscr{X})(x \in \mathscr{X}, \lambda \in \mathbf{C} \setminus \{0\})$;

(iv) if T is m-decomposable ($m \geqslant 2$) then $\sigma(x; T, \mathscr{X}) = \bigcap\{F \in \mathrm{Cl}(\mathbf{C}^n): x \in X_T(F)\}$ ($x \in \mathscr{X}$) and $X_T(F) = \{x \in \mathscr{X}: \sigma(x; T, \mathscr{X}) \subset F\}$.

Not all of the usual properties enjoyed by operators with the s.v.e.p., however, remain true in the multivariable case. For instance, if an operator T has the s.v.e.p. and $\mathscr{M} := \{x \in \mathscr{X}: \sigma(x; T, \mathscr{X}) \subset F\}$ is closed (for some $F \in \mathrm{Cl}(\mathbf{C})$), then $\mathscr{M}$ is spectral maximal and $\sigma(T|_{\mathscr{M}}) \subset F$. J. Eschmeier has recently used the fact that the above result is false for $n = 2$ to produce an example of a commuting pair of decomposable operators which is not decomposable; of course, if $T = (T_1, \ldots, T_n)$ is decomposable, so is each T_i ($i = 1, \ldots, n$). (As is usually the case in multivariable spectral theory, if a property P holds for an n-tuple then P holds for each coordinate, but the converse is rarely true (e.g., subnormality, similarity, decomposability); in the situation at hand, the projection property for the Taylor spectrum allows one to get a spectral capacity for the individual operators from that of the n-tuple.)

We have mentioned at the beginning that there are basically two directions in which spectral operators can be generalized, and in the previous paragraphs we have discussed one approach. We shall now concern ourselves with the nonanalytic functional calculus perspective. First, one needs to establish some basic conditions on the algebras of functions that can be admitted. An algebra $\mathcal{U}$ of complex-valued functions on a closed set $\Omega \subset \mathbf{C}^n$ is called admissible if $\mathcal{U}$ satisfies (1) $1 \in \mathcal{U}$ and $z_1, \ldots, z_n \in \mathcal{U}$; (2) $\mathcal{U}$ is normal, i.e., for every finite covering $\{U_1, \ldots, U_k\}$ of $\overline{\Omega}$ there are functions $f_1, \ldots, f_k \in \mathcal{U}$ with $0 \leqslant f_j \leqslant 1$, $\operatorname{supp}(f_j) \subset U_j$ $(j = 1, \ldots, k)$ and $f_1 + \cdots + f_k = 1$ (Existence of partitions of unity); and (3) for every $f \in \mathcal{U}$ and $\lambda \notin \operatorname{supp}(f)$ there exist $f_1, \ldots, f_n \in \mathcal{U}$ such that $\sum_{i=1}^n (\lambda_i - z_i) f_i(z) = f(z)$ for all $z \in \Omega$. (Example. For Ω compact, $\mathcal{U} = B(\Omega)$, the algebra of bounded Borel functions on Ω, is admissible.)

If $\mathcal{U}$ is an admissible algebra on $\Omega \subset \mathbf{C}^n$, an n-tuple T is called *$\mathcal{U}$-scalar* if there exists a unital homomorphism Φ: $\mathcal{U} \to \mathcal{L}(\mathcal{X})$ such that $\Phi(z_i) = T_i$ $(i = 1, \ldots, n)$. It is then not hard to prove that if T is $\mathcal{U}$-scalar then $X(F) := \bigcap\{\ker \Phi(f) \colon f \in \mathcal{U}, \operatorname{supp}(f) \cap F = \varnothing\}$ defines a spectral capacity for T, so that T is decomposable. For a relatively open set $G \subset \Omega$, Φ is said to be null on G if $\Phi(f) = 0$ for every f such that $\operatorname{supp}(f) \subset G$. Then $\operatorname{supp} \Phi := \bigcap\{F \in \mathrm{Cl}(\Omega)\colon \Phi \text{ is null on } \Omega \setminus F\} = \sigma(T, \mathcal{X}) = \sigma_{(T)'}(T) = \sigma_l(T) = \sigma_r(T)$; moreover, $\operatorname{supp} \Phi(\cdot)x = \sigma(x; T, \mathcal{X})$, for all $x \in \mathcal{X}$. A $\mathcal{U}$-scalar n-tuple for which $\mathcal{U}$ is an inverse-closed subalgebra of $C(\Omega)$ (the continuous functions on Ω) behaves very much like a spectral operator:

(a) $M_{\{\Phi(f): f \in \mathcal{U}\}} \cong \sigma(T, \mathcal{X})$ and $(\Phi(f))\hat{} = f|_{\sigma(T, \mathcal{X})}$;

(b) $\sigma(\Phi(f), \mathcal{X}) = f(\sigma(T, \mathcal{X}))$;

(c) for every $f \in \mathcal{U}$, $\Phi(f)$ is decomposable and $X_{\Phi(f)}(F) = X_T(f^{-1}(F))$, $F \in \mathrm{Cl}(\mathbf{C})$;

(d) Φ is unique up to spectral equivalence, i.e., for every other unital homomorphism Ψ: $\mathcal{U} \to \mathcal{L}(\mathcal{X})$ such that $\Psi(z_i) = T_i$ $(i = 1, \ldots, n)$, one has that $\Phi(f)$ and $\Psi(f)$ are spectrally equivalent (= quasinilpotent equivalent in the sense of [2]) for every $f \in \mathcal{U}$.

An important special case is $\mathcal{U} = C^\infty(\mathbf{C}^n)$. A C^∞-scalar n-tuple T is said to be (generalized) scalar if the homomorphism Φ is continuous; Φ is then called a spectral distribution for T. Spectral distributions (which are not necessarily unique) for (generalized) scalar n-tuples T are always extensions of the (coherent) functional calculus associated with T. Also, if Φ is a spectral distribution for the scalar n-tuple T, then $\Phi(f)$ is scalar for all $f \in C^\infty(\mathbf{C}^n)$. Another important choice for $\mathcal{U}$ is $B(\Omega)$, the algebra of bounded Borel functions on Ω. For many $B(\Omega)$-scalar n-tuples T, the spectral capacity X_T can be described in terms of a (projection-valued) spectral measure E defined on the Borel subsets of Ω by $X_T(F) = E(F)\mathcal{X}$, for $F \in \mathrm{Cl}(\mathbf{C}^n)$. If T is a commuting n-tuple and E is a spectral measure such that $T_i E(F)\mathcal{X} \subset E(F)\mathcal{X}$ $(i = 1, \ldots, n)$ and $\sigma(T, E(F)\mathcal{X}) \subset F$ (for all $F \in \mathrm{Cl}(\mathbf{C}^n)$), then T decomposes uniquely as $S + Q$, where $S_i = \int \lambda_i \, dE(\lambda)$ and Q_i is quasinilpotent $(i = 1, \ldots, n)$; moreover, $S_1, \ldots, S_n, Q_1, \ldots, Q_n$ mutually commute. (Example: An n-tuple T similar to a commuting n-tuple of normal operators on a Hilbert space $\mathcal{H}$ has a spectral measure E on $\sigma(T, \mathcal{H})$ with the above properties.)

The construction of the analytic functional calculus for several commuting operators and the axiomatic study of spectral decompositions both in the multivariable case and for single operators constitute the two central parts of this book. The author has made a commendable effort to present the Cauchy-Weil integral for commuting systems on Fréchet spaces with very little use of homological algebra, thus making the construction more accessible to the nonexpert, although there is naturally a certain loss of elegance. For Hilbert spaces, the connection between the Cauchy-Weil integral and the functional calculus given by the Martinelli kernel is also well established. A number of results concerning parametrized chain complexes, Fredholm complexes, and applications of Taylor's theory on Banach spaces have been stated and proved in the context of Fréchet spaces. All the needed facts from the theory of several complex variables are either proved or have adequate references. In this aspect the book is quite self-contained, and it includes a wide spectrum of basic topics, ranging from the definition of dimension for metric spaces to a proof of Cauchy's formula at infinity.

A whole chapter is devoted to a systematic presentation of examples, counterexamples and applications, related both to the notion of decomposability and to joint spectra; in most cases, full details are given. They supplement the material presented in Chapters III and IV and enhance the importance of decomposable systems in operator theory. A discussion of the relation between existence of a nonanalytic functional calculus for an operator and growth conditions on its resolvent is also included, following in general terms the scheme that (for spectral operators) Dunford presents in [3, §4].

The book under review is an English version of the author's monograph, *Calcul funcţional analitic multidimensional* (in Romanian), published in 1979. The author has made a number of improvements, additions and corrections; new results have been incorporated either in the text or as part of the *References and Comments* section at the end of each chapter. There are very few typographical errors, most of them minor ones; we mention, however, four that may cause some confusion: in the statement of Lemma IV.1.20, "$\overline{U} \cap K = \varnothing$" should read "$\overline{U} \cap \mathrm{D} = \varnothing$"; in the statement of Corollary IV.7.11, "$F \in \mathrm{Cl}(\mathbf{C}^n)$" should read "$F \in \mathrm{Cl}(\mathbf{C})$"; in the proof of Lemma IV.1.19, $F = F_1 \cup F_2$; and in Definition IV.1.5(2), each F_j must be assumed to be in $\mathscr{F}$. The *Subject Index* refers a word or notion to a section number rather than a page number; there is at times too much dependence on the material of Chapters I and II, and the *Notation Index* helps alleviate that to a great extent (e.g., a reader who wants to skip those chapters and turns to p. 68, line +7 ("For every $T \in \mathscr{C}(X)\ldots$") will find the definition of $\mathscr{C}(X)$ nowhere in Chapter III; $\mathscr{C}(X)$ is actually defined on p. 9, and the notation index includes that information.)

As the author states in the preface, the book "is intended to be read by specialists in operator theory and graduate students who want to go more deeply into the matter." A potential reader must have a solid knowledge of the basic tools of functional analysis and operator theory. Assuming this, the book does provide enough material for a topics course. The chapter on spectral decompositions contains a wealth of results that make the treatment quite comprehensive and detailed. m-decomposability, however, has been replaced by a more general notion, $(\mathscr{F}, \varphi, m)$-decomposability, where $\mathscr{F}$ is a pseudoring of closed sets in a topological space Ω, φ: $\mathscr{F} \to \mathrm{Cl}(\Omega)$, and $m \geqslant 1$; although no motivation is given for such an increase in generality, its usefulness later becomes apparent (for instance, in the study of real decomposable operators). (To prepare the reader for Chapter IV, E. Albrecht's survey article **[1]** may prove to be quite helpful.) In the chapter on multivariable spectral theory, the author focuses attention almost exclusively on Taylor's notion of joint spectrum (W. Zelazko's axiomatic approach is, however, treated at some length), and as a result of that, many important related topics are skipped. This is quite understandable if one recalls that the main purpose of the chapter is the detailed construction of the analytic functional calculus as an indispensable tool for the theory of spectral decompositions. A similar assessment can be made regarding the bibliography, which, although substantial, is "far from being extensive. It contains mostly works that have [strict connections with the] text and only a few works of general interest."

Since the landmark monograph of Colojoară and Foiaş, the theory of spectral decompositions has made a significant progress. The author, as one of the chief contributors to these developments, has presented here a state-of-the-art account. In the last two years, however, various contributions have been made, and we would like to mention the following: J. Eschmeier's local analytic spectral theory has been refined and improved by M. Putinar and the author; M. Putinar has developed a sheaf-theoretic model for commuting n-tuples, which seems to provide a better generalization of decomposability, and is relevant to the study of generalized Bergman kernels (for certain subnormal n-tuples, the obstruction to being decomposable can be measured in terms of the cohomology of their sheaf models); E. Albrecht and R. Mehta have considered a "spatial" local spectral theory in the Calkin algebra (Toeplitz operators with continuous symbols are generally not decomposable, despite the fact that modulo the compact operators they have a rich functional calculus, and this work is aimed at understanding that phenomenon); the author has begun a study of the decomposition theorem for analytic operators (operators from the space of entire analytic functions on $\mathbf{C}^n$ to a Fréchet space $\mathscr{X}$); and M. Putinar has shown that a hyponormal operator is subscalar of order 2. Concerning multivariable spectral theory, recent developments include the following: an index theory for Fredholm n-tuples has been independently obtained by R. Carey and J. Pincus, and by M. Putinar; the spectral and Fredholm theory of generalized derivations has been described in terms of joint spectra (L. Fialkow, A. Carrillo-C. Hernández, and the reviewer); an L-shaped domain has been used to provide an example of a non-type I C^*-algebra of Toeplitz operators (P. Muhly and the reviewer); subnormal and Toeplitz n-tuples have been investigated by M. Putinar, N. Salinas, the reviewer and others; M. Cowen and R. G. Douglas have extended to several variables their work on the classes $B_n(\Omega)$; and a systematic study of closures of joint similarity orbits has been initiated by D. A. Herrero and the reviewer. The theories of spectral decompositions and of joint spectra will continue to actively interact with complex geometry and with the theory of several complex variables.

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RAÚL E. CURTO

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Computation with recurrence relations, by Jet Wimp, Applicable Mathematics Series, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1984, xii + 310 pp., $50.00. ISBN 0-273-08508-5

Recurrence relations occur in a variety of mathematical contexts. They connect a set of elements of a sequence of some type, usually either numbers or functions, such as coefficients in series expansions obtained by undetermined coefficients, moments of weight functions, and members of families of special functions. They can be used either to define the sequence or to produce its elements.

They lead to concise algorithms which are useful for either manual or automatic calculations and can allow great economy in tabulation or approximation. Algorithms based on recurrences are particularly useful for automatic computers because of the compact programs to which they lead, with concomitant economies in memory requirements and in error elimination.

Serious difficulties may be encountered, however, when inexact arithmetic or initial values are used. For example, the modified Bessel functions of the first kind, $I_n(x)$ satisfy the recurrence:

(1) $$y_{n+1}(x) = -(2n/x)y_n(x) + y_{n-1}(x).$$

For $x = 1$, they are positive for all n, and decrease monotonously toward 0 as n increases. Using values for $I_0(1) = 1.266065878$ and $I_1(1) = 0.5651591040$, correct to 10 significant digits, and computing $I_2(1), I_3(1), \ldots$ by (1), we find

n	$I_n(1)$	n	$I_n(1)$	n	$I_n(1)$
0	0.1266065878 (+1)	1	0.5651591040 (00)	2	0.1357476700 (00)
3	0.2216842400 (−1)	4	0.2737126000 (−2)	5	0.2714160000 (−3)
6	0.2296600000 (−4)	7	−0.4176000000 (−5)	8	0.8143000000 (−4)
9	−0.1307056000 (−2)	10	0.2360843800 (−1)	11	−0.4734758160 (00)
12	0.1044007639 (+2)	13	−0.2510353092 (+3)	14	0.6537358115 (+4)

These absurd numerical values are caused by instability in using this recurrence for $I_n(x)$ for increasing n. Such difficulties are familiar to numerical mathematicians in many contexts, although they may not be as generally recognized as would be desirable.