Then follows the interesting story of how Dirichlet, a young student in Paris in 1825, tried his hand on Fermat's equation with exponent 5. He used an identity known since Euler's days and the infinite descent to show that the equation has no solution where one of the numbers is a multiple of 10. "This is where Legendre, then well over 70 years old, stepped in. After presenting Dirichlet's paper to the Academy in July 1825, it took him only a few weeks to deal with the remaining case." "As to Dirichlet, he was soon to take his flight and soar to heights undreamt of by Legendre.."

One of Legendre's influential contributions was the treatise on numbers which he prepared for more than thirty years. "He sought to give a comprehensive account of number theory, as he saw it at the time, including, besides his own research, all the main discoveries of Euler and Lagrange, as well as numerical evidence (in the form of extensive tables) for many results whose proofs he felt to be shaky."

The *Théorie des Nombres*, published in 1830, is the final form given to two previous editions, appropriately called *Essais*. Yet, "by then, as his younger contemporaries well knew, Gauss's *Disquisitiones* had made it almost wholly obsolete."

An indispensable part of Weil's book is the long series of appendices attached to the three main chapters. Their purpose is to show, from a modern point of view, how to consider certain classical questions, to indicate developments of importance originated in the ideas of that period, but sometimes also to give proofs of results described in the main text. Thus, we may read an illuminating appendix under the title "The Descent and Mordell's Theorem", another about "The Addition Theorem for Elliptic Curves", or also "Hasse's Principle for Ternary Quadratic Forms", etc.

Here I reach the point when it is appropriate to refer to the physical characteristics of the book. Should I say that it is a medium-sized volume, well bound and pleasantly printed, with large size type, greatly facilitating the reading? Should I add that it is well organized, has good indices, and no misprints? I just want to say that the hand holds it well, and does not wish to let it go.

Professor Weil, hear as a distant echo from younger days: Rico é o seu livro que nos revela uma gloriosa exploração intelectual pelos verdadeiros heróis.

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Fundamentals of generalized recursion theory, by M. C. Fitting, Studies in Logic and The Foundations of Mathematics, Volume 105, North-Holland Publishing Company, Amsterdam, 1981, xx + 302 pp., $63.75 U.S./Dfl. 150.00.

0. The time seems ripe for a broad look at generalized recursion theory (g.r.t.) against the background of ordinary recursion theory (o.r.t.), with so-called relativizations of o.r.t. at the border. O.r.t. started some 50 years ago. The more infinitistic part of g.r.t., the main subject of the book under review, goes back almost 30 years, with a kind of half-time score on pp. 139–198 of *Logic* '69 **[GY]**.

Below, the emphasis will be shifted; away from differences within the—realistically speaking, quite narrow—logical tradition with its flashy metatheorems to its differences from the broad mathematical tradition. Correspondingly, *combining* elements of logic with specific subject matter will be opposed to elaborating *undiluted logical notions* relentlessly. Implications of these points are here set out for o.r.t. and the passage(s) to g.r.t., but they extend to most other branches of logic. The same applies to the following elementary distinctions, which will be needed throughout the review.

1. Fundamentals and generalizations. Notions can be fundamental or, equivalently, basic in the sense of striking our untutored attention. Any uses they may have at an early stage will naturally require only quite elementary properties of such notions. Almost as a corollary, elaborations of them are liable to reach quite soon the point of diminishing returns (for the uses in question). These truisms are illustrated very well by the basic notion of natural number and its uses for counting or ordering finite sets; most dramatically by reference to elaborate "lower" arithmetic of perfect or amicable numbers, comparable to certain labor-intensive parts of o.r.t. Less obviously, such elaborations may draw attention away from other mathematics that is much more effective in the broad area of those original uses. Counting provides yet another memorable illustration. Generally—with such obvious exceptions as counting primes or other darlings of the queen of mathematics—Higher Arithmetic has little bearing on what or how to count; Higher Statistics often has more bearing, for example, when the collections involved are large, and function theory helps count zeros of the zeta function.

Now, at least in science there is a second sense of basic. It applies to things and properties that are discovered, usually by long experience, to lend themselves to extended or even systematic study. They may be abstract properties of the first kind of basic notion, as in basic number theory and the natural numbers above or the—by p. 194, 1.16–17 of **[C1]**—less basic rational numbers. Abstract properties are usually formulated axiomatically. Instant choices of such axiomatizations are common in the logical tradition for the sake of the logical ideal of axiomatic precision. The distinction between the two kinds of basic notions has some implications for generalizations.

Traditionally, the first kind of basic notion has often been generalized in the crude sense of its original domain of definition being extended. This can be useful even if no recondite mathematical properties are preserved, only elementary ones being needed in the first place. If Cantor's so-called generalizations of numbers are viewed as such extensions to infinite sets and well orderings and not as competing with Higher Arithmetic (say, of ideals or algebraic number fields, preserving decomposition properties of primes and not only algebraic identities), then their marginal utility is quite reasonable; not, of course, the fuss made about them, let alone dubious doubts about their legitimacy.

For the second kind of basic notion the more modern axiomatic style of generalization is popular. If, as at the end of the last paragraph but one, the notion is an axiomatically formulated abstract property, it *is* a generalization, the defining axioms *being* the properties to be preserved. The colorless term 'generalization' tends to hide the implications of this axiomatic style for a so-to-speak mathematical proof analysis, here opposed to logical proof theory taken up in §7. A particular choice of abstract properties constitutes a scheme for breaking up arguments or, generally, situations into a few elements, and hence easy to take in. These elements are described in terms of a few basic structures, and hence easy to remember; with improved reliability as a bonus. Without exaggeration: such choices of generalization express discretely what would coarsely be described in terms like 'important', 'relevant', or 'essential' (for the body of mathematics being generalized). Viewed this way instant choices merely express a lack of understanding of the amount of experience needed to spot, and, above all, test generalizations. All this fits in with the relative maturity of g.r.t. mentioned at the outset.

2. O.r.t.: background. Its elementary part concerns domains D consisting of words on a finite alphabet, for example, ω: the natural numbers generated from 1 by the successor or 'concatenation'. O.r.t. studies recursively enumerable (r.e.) subsets of D, partial and total functions: $D \mapsto D$, and simple relations between these objects. Incidentally, though mathematically trivial the extension, from ω, to the domains D above, is involved in *nonnumerical* data processing, a substantial part of the industry.

The first, still memorable descriptions of the objects treated in o.r.t. were in terms of (a) simple properties of models for formal systems, and of (b) the perfect computer. Of course, conversely (a) and (b) can be described in terms of o.r.t. (though this is anathema to the foundational urge).

(a) Here the self-explanatory key words for recursive sets are: now *invariantly definable*, orginally *formal entscheidungsdefinit* (in Gödel's incompleteness paper **[G1]**); cf. Note 1.[1]—For an arbitrary subset X of D, so-called X-relativized o.r.t. concerns formal systems to which the *diagram* of X is added, a familiar object of elementary model theory.

(b) The perfect computer requires less background than (a) above, and so provides a more widely accessible description of o.r.t. It also seems more popular, especially when presented as an idea(lization)—and almost certainly *the* idealization if one knows nothing—of genuine computers; perhaps, how *Simple Simon* or *der kleine Moritz* imagines them.

Be that as it may, the perfect computer embodies a novelty not even suggested in the old literature on, say, numerical analysis. Now rules are applied to other rules, and occasionally to themselves; in trade jargon, the same codes are used for instructions and their arguments. According to his autobiography **[Z]** this possibility escaped Zuse who patented a sound relay computer in 1937; but not von Neumann, whose insight has been used for efficient software, at least, ever since the hardware passed certain thresholds of

[1] The notes, at the end of the review, are intended for readers with some logical background or, at least, interest.

memory size and speed. Facile questions of cause and effect aside—here, and below whenever as little is known about the phenomenon involved as here about the genesis of bright ideas—it is certainly *satisfaisant pour l'esprit* that von Neumann was familiar with, and impressed by, an objectively related device in **[G1]**.

The use of that insight *combines* a little o.r.t. with higher (electronic) technology, an instance of a theme in §**0**. To be effective this use has to respect the limitations of the hardware, and the needs of the computations in question; key words: choice of parameters or, in fancy language, of data structures.

Logicians see the novelty above very differently; as a *universal principle* for generating *all* objects of o.r.t. from a few instances, for example, as rule S9 in **[K2]** (but overshadowed by the material there on extending o.r.t. to all finite types over ω; cf. Note 9b). Naturally, this principle is valid only if the limitations mentioned in the last paragraph are neglected. So any glamor it may have draws attention away from the need for respecting them.

At this point it is necessary to say a word about current complexity theory with its would-be revolutionary separation P/NP (polynomial versus nondeterministic polynomial growth) in place of: recursive/nonrecursive. By implication it regards any reminders about the perfect computer, such as those above, or generally about the computational side of o.r.t., as obsolete. They may be. But as with many popular revolutions the coarsest elements of the old order remain in the new. Specifically, like old o.r.t., complexity theory concentrates on logical classes of problems; cf. Note 2a. The bounds obtained simply show that those classes do not lend themselves to algorithmic treatment, and flashy theorems about 'large' bounds have drawn attention away from more successful choices of problems already in the literature. Incidentally, the significance (in the here relevant statistical sense) of many *average* complexity results has not been established in the currently fashionable probabilistic literature, especially, for those logical classes. What is needed here is, first, some plausible distribution of problems within those classes, and then an estimate of *mean deviations*. Nothing of the kind is to be found in the bulk of the literature.

Granted all this the brutal fact remains that, in the area of computation, elaborations of o.r.t. tend to share the weakness of Higher Arithmetic for counting mentioned in **1**.

How then is o.r.t. to diversify? A hint comes from past experience with the idea of perfect liquids, notoriously imperfect except for a very few corners of hydrodynamics. Progress was made by *shifts of emphasis* away from the original context. The two-dimensional motion of such liquids provides a valid description of—not merely, as is sometimes said, a metaphor for—the notion of function of a complex variable. The latter is firmly established in mathematics, even used in parts of mathematical physics, but just not primarily in successful hydrodynamics.

Shifts of emphasis are also the subject of pious talk about preestablished harmony, but with a difference. Stressing continuity rather than change, it hides the progress, and, above all, the imagination involved in successful shifts. The harmony between an early idea and its successful offshoots can be hard to detect behind the intellectual pollution on the way. Thus many old-fashioned Tripos problems, without mathematical or physical interest, created an illusion of relevance to hydrodynamics, simply by being formulated in the language of perfect liquids.

3. O.r.t.: philosophical progress through mathematical theorems, in particular, without illusions about the perfect computer. As in **0** and **2**, o.r.t. is combined with (higher) technology, this time from group theory and early number theory, knowledge of these subjects being the price for progress.

Finitely generated groups G. About 25 years ago, Higman discovered [**H**] that such a G can be embedded in a finitely presented group iff its word problem, WP_G, is r.e., where $WP_G = \{w\colon w \in G, w = 1\}$. It is easy to exhibit infinitely presented G such that WP_G is r.e., but not recursive. As a corollary there are finitely presented G with a recursively undecidable WP. In contrast to this corollary, Higman's theorem is not spoilt by defects of the perfect computer as an idealization of genuine decision procedures (by electronic, biological or other analog computers; cf. Note 3). Imagination was needed for the few rewarding twists on Higman's results, in particular, by A. Macintyre and M. Ziegler for existentially closed groups, requiring more background than [**H**]; for documentation, cf. [**ABH**].

Diophantine sets D_n are $\{a\colon a \in \omega, \exists x_1, \ldots, \exists x_n\ (P = Q)\}$ where P and Q are polynomials in $x_1, \ldots, x_n$ (over ω) with the parameter a and coefficients in ω. During the 70s it was shown that, for $n \geqslant 9$, r.e. = (the class of all) D_n. The reduction, to $n \geqslant 9$, from Matyasevic's original bound around $n \geqslant 256$, involved not only number-theoretic, but also recursion-theoretic coding tricks. This discredits the usually unqualified ideology of an index-free exposition of o.r.t., popular in the 50s, but also on p. ix, l.–2 of Fitting's book; cf. also Note 4.

Probably no single result about *all* diophantine sets or equations is as memorable to the general mathematical public as the description above in terms of o.r.t. Also, diophantine definitions of the set of primes have appeal, and, presumably, will some day find some use in some corner of the subject. But there are dirty spots on this glittering surface.

How rewarding an object of study is the totality of the D_n, even for $n = 9$? As far as mechanical decidability (of a D_n being empty) is concerned, it demonstrably is not. The next question is more specific. It concerns the passage between number-theoretic properties (of D_n) and those (of r.e. sets) prominent in o.r.t. Are the formal equivalents comparably adapted to the other domain? For example, maximality of r.e. sets, modulo finite sets, is a favorite of o.r.t. Does any number-theoretic equivalent contribute to some prominent problem in the number-theoretic tradition? Conversely, one wonders about equivalents in o.r.t. of those geometric classifications of diophantine equations which have genuinely revolutionized this subject. And, as always, there are questions about questions, particularly since traditions are not sacrosanct. Are there not blind spots, in o.r.t. or traditional number theory or both, that would be removed by pursuing those equivalences more vigorously, thus leading to new rewarding notions and problems?

Experience shows that all-purpose questions such as those above tend to answer themselves with better knowledge of the subject. This should be enough for the present review. But readers interested in popular views of logic—especially at the two extremes of enthusiasm and distrust—are recommended to look at the panaceas, for those and related questions, implicit in the logical literature.[2]

It remains to note that Ershov sets out in **[E]** how o.r.t. can be applied outside its original domain; 'in principle', to any enumerated structure (with some hints on appropriate structures; but cf. Note 9a). At least so far, like **[H]** the most convincing applications have used only elementary parts of o.r.t.

4. O.r.t.: mathematical proof analysis. Since such analysis of elementary matters tends to degenerate into pedantry, the more advanced part of o.r.t. will be considered. By common consent it is dominated by so-called priority arguments. They involve a kind of nested diagonalization, elaborating the wide use of simple diagonalization in elementary o.r.t., but also elements strongly reminiscent of other parts of mathematics and logic. For example, balancing out of terms for tricky convergence proofs in analysis or for proofs of measure $< \varepsilon$ is such an element, and above all Ackermann's modification of Hilbert's ε-substitution method where the order of priority is determined by the rank of an ε-term, and its value can change only finitely often in the process; cf. top of p. 29 in **[HB]**. In any case nothing below assumes that the style of those (priority) arguments is unique to o.r.t., though it may well appear so to those dedicated wholly to this subject.

Many results proved by priority arguments concern some kind of *degree* or other in the sense of o.r.t. This sense is conveyed quite well by a parallel from very early number theory, in Euclid's Book X, recently discussed in this Bulletin **[Kn]** under the heading: *La croix des mathématiciens.*

Degrees of Book X: classification of irrationals. Geometrically simply defined numbers, like $\sqrt{2}$, had been discovered to be irrational. What should one do with them? Book X classifies them; of course in then-familiar terms, now called 'Euclidean'. The notion of degree involved is strictly between rational and algebraic dependence. By the way, the question above remains long after the Pythagorean slogan, about rational numbers being the measure of all things, has been discarded.

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[2]A principal claim *behind* the austere language of logical texts is that the particular generality of logical notions provides the following recipe for the systematic pursuit of knowledge. One begins with logical classifications, defined for anything under the sun, thus not presupposing specific experience. As the latter accumulates one refines them; tacitly, in contrast to replacing them by others that cut across them; cf. the end of Note 3 on the glamor of logical languages, and Note 2a on examples of incomparable classifications. The claim *responds to a popular demand*, usually expressed in fleeting complaints (while in the brutal form above both the claim and the demand would be brushed aside by any experienced scientist). For example, Cassels complains in **[C2]** about the unsystematic character of the body of number theory that has accumulated. This certainly neglects the distinction in **1**; but perhaps even the noteworthy fact of experience that number theorists like himself have the capacity of handling quite a battery of notions, by proper use of sound intellectual reflexes as it were. Popular views of logic are surely determined much more by such broad considerations as in this footnote than, for example, by the results stressed in the review itself.

Degrees of o.r.t. classify nonrecursive sets (of words). Here the starting point is Gödel's discovery of such sets that are logically simply defined; for example, the r.e. sets of theorems provable in *Principia Mathematica* (and related systems in the sense of [**G1**, p. 190]) and in elementary logic (Satz 9 and 10 of [**G1**]). Of course, the classifications use some kind of recursive operations. Here the discarded slogan goes back to Hilbert, about formal methods 'measuring up' to all problems. They don't; by [**G1**] not even to the particular problems of deciding formal derivability in the systems mentioned above.

The broad schemes proposed in o.r.t. for classifying all, or at least the r.e. nonrecursive, sets are unconvincing enough for its degree theory to have been called *la croix des logiciens*. What is one to do with all those degrees? Naturally, as with irrationals, some special cases are unproblematic, such as Ziegler's degrees in **3** used to state facts about existentially closed groups. However, as far as a general theory in the forseeable future is concerned, the parallel with Book X is daunting.

Time scales may have changed since Euclid. But there is such a thing as the absolute level of imagination that went into the shifts from the degrees of Book X to its modern heritage: diophantine approximations and measures of irrationality, with implications for diophantine equations. More recently there have been shifts from the brutal classification of algebraic numbers by (their algebraic) degree to less hackneyed selections; perhaps, most simply in [**B**]: $\sqrt[n]{\alpha}$ for any algebraic α and $n \in \omega$, $n \neq 1$ and $\alpha \neq 1$. Even when the high spots, for example, in [**ABH**], are counted, nothing in o.r.t. approaches the philosophical detachment from the original set-up that was so essential for progress in the parallel from number theory.

Some progress in proof analysis of degree theory becomes visible when expectations are lowered drastically. As so often, one finds results stated in terms of o.r.t., but easily seen to be corollaries of much more general facts. The principal result on degrees picked out in Fitting's book has been known for some 20 years to be of this kind; cf. Note 5a. Less formal shifts of emphasis have gone in the opposite direction as it were: What more do we know when we do use hypotheses or styles of arguments that are not (logically) needed for the result stated? To summarize such additional knowledge a new theorem is usually needed, possibly stated in terms of new concepts; cf. Notes 5b and 5c. All things considered, including the warning about instant generalizations in **1**, it is probably premature to analyze degree theory axiomatically. For the time being 'straight' expositions such as [**Sh** and **Y**], perhaps supplemented by a couple of quotable results like those of [**HS** or **Sho**] on (elementary theories for the order of) r.e. and Δ_2^0-degrees, seem more rewarding than, say, axioms for so-called applicative structures; cf. Note 6.

5. G.r.t.: the old-fashioned style. In contrast to what has just been said about axiomatic generalizations of o.r.t., several of its extensions have done well; at least, in logic and set theory, including descriptive set theory. We begin with the oldest and mildest g.r.t.

Relativized o.r.t. (a) Already its simplest form is an efficient tool for sharpening quantitatively a, if not the, main discovery about elementary logic (back in the 30s). Contrary to early ideology about formalization, most results hold for arbitrary sets of axioms, and not only for formal systems with their r.e. sets; cf. Note 1a (ii) for a typical use of relativized o.r.t. In other words, relativized o.r.t. helps to clean up the logical pollution of literal o.r.t.

(b) For $X \subset \omega$, well orderings (on ω) defined in X-relativized o.r.t. determine a class of order types, that is, of countable ordinals. The supremum is called ω_1^X. These in turn determine segments, with useful closure properties, of certain hierarchies of—definitions of—sets. They were called 'predicative' or 'ramified' at the turn of the century, by Poincaré, resp. Russell, later 'constructible' (by Gödel **[G2]**), later still 'hyperarithmetic' (at least, when restricted to a certain segment) by Kleene **[K1]**. The ups and downs in the level of sophistication displayed were great, and the close relations between the notions themselves were noticed relatively late; cf. the review of **[K1]**, and pp. 105–106 of **[S]**. Readers can get a good idea of the definitions involved from old-fashioned introductions of closures in algebra by successive adjunction of roots of polynomials; for example, real and algebraic closures or, to go back to Book X, Euclidean closures. The principal difference is that the algebraic adjunctions depend only on finitely many arguments, the coefficients of the polynomials considered, while in general the set-theoretic ones do not. The exposition in **[G2]**, with its seven adjunction operations corresponding to the build-up of logical formulas (and accumulation) is better suited to this parallel than **[K1]** which adjoins to any element its jump and the infinitely many objects recursive in it (apart from diagonalization).

Before answering the inevitable question, what to do with those ω_1^X, another description of them is worth mentioning; at least, for those familiar with the foundational literature at the turn of the century. One of its idea(lization)s of the *perfect definition* was neatly conveyed by Poincaré and later made impeccably precise: the meaning of such a definition must not be changed by extending the universe (of sets; and here it matters little if we think of sets-of-numbers or sets-of-sets-of...). As an analysis of the possibilities for valid or precise definitions, Poincaré's idea is probably even less perfect than the perfect computer and the perfect liquid are in their domains; an insight which, by the way, constitutes philosophical progress in the so-called theory of definitions. But around 1960 the idea served quite well for deriving mathematical properties of the ω_1^X and of the segments determined by them, when the set X and, tacitly, ω itself were thought of as 'perfectly defined'. The ω_1^X will turn up again in Note 9b. But the following description, and hence use, of these objects seems of wider interest.

G.r.t. derived from the model-theoretic description of o.r.t. in **2**a. Now some models are privileged, for example, those in which the range of some variable is prescribed (to be **Z** or **Q** in so-called ω-models). By around 1960 the definitions indicated in **2**a had been transferred verbatim, with the bit of extra care specified in Note 1b. Well orderings defined in relativized o.r.t. come in through a, generally infinitary, calculus for validity in the privileged models, corresponding to ordinary predicate calculus in o.r.t.; cf. Note 1a(i). Work after 1960 fitted in with the state of model theory for elementary logic at that time. In particular, the latter was then dominated by the compactness theorem, originally called finiteness theorem (by Malcev). So an analog for ω-logic was looked for, and found; but in contrast to Note 1a(ii) and (a) above, only for r.e. sets of axioms in the generalized sense.

These simple ideas, originally circulated under the trade name 'metarecursion', were later spread out in volumes on so-called admissible sets; cf. Note 7a. But it is fair to say that the original hopes have not been fulfilled at all. In retrospect they are seen to have involved a simple, but, at least in logic, common oversight. Many familiar structures, actually throughout mathematics, are the ω-models of quite simple axioms in logical language; for example, vector spaces over **Q** (while vector spaces over arbitrary fields are the corresponding ordinary or 'general' models). This alone is no guarantee that a theory of arbitrary ω-models will tell us what we want to know for any one of them! As noted in **1**, likely candidates have to be discovered; for example, p-adic fields in ordinary model theory—which has not been equally successful in the rest of mathematics.

If the hopes had been confined to some corner of mathematics such as descriptive set theory, later developments would have more than lived up to such early items, of g.r.t., as the quantitative refinement for the theorem of Cantor-Bendixson; cf. Note 7b. If ever significant relations were to be discovered between that corner and broader areas of mathematics, even such refinements—or, more likely, variants depending on those hypothetical relations—might acquire a bit more interest. We can end this section on a more positive note. *Within* the 'pure' theory of ω-models the contribution of g.r.t. is evident enough by comparing the clumsy literature in the 50s with modern expositions.

6. G.r.t. and fine structure theory. G.r.t. looks much better in the light of the following parallel which is outright *satisfaisant pour l'esprit*. It concerns two developments in the 60s.

One side of this parallel is the work—with its lively exposition—by Sacks. He and his students developed degree theory for g.r.t. related to the hierarchies in **5**; often using substantial machinery different from material known in o.r.t. (in contrast to the simple results of **5**, proved or at least provable in a couple of lines). So to speak conversely, different proofs of the same theorem in o.r.t., for example, on maximal sets were generalized to different theorems in g.r.t. In due course, (closure) properties of ω were discovered that are shared by some, but not all ω_1^X, and determine whether or not a theorem of o.r.t. generalizes; again, maximal sets provide memorable examples. In short, some of the many ordinals between the first 2 infinite cardinals began to look promising.

The other side of the parallel is the growth of confidence, among other logicians, in fine structures, that is, in details of the hierarchies in **5**; a high spot is Jensen's work on the hierarchy L itself [J]. In the early 60s most set theorists, including model theorists involved with infinitely long formulas, thought of ordinals between cardinals as a grubby business; with a few exceptions cited in **5**. By the end of the 60s there was confidence, in the same circles, in intermediate ordinals; not merely for defining hierarchies of course, but as a subject for theorems or, at least, as a means for more delicate arguments by induction.

Questions about any—conscious or unconscious—'mechanism' of discovery aside, [J] presents itself as a combination of ideas used in g.r.t., including suitable—if not yet: master—codes, toys from logical proof theory such as Bachmann's hierarchies with coherent fundamental sequences, and other things, partly to be found in Jensen's (earlier) *Habilitationsschrift* on admissible sets. Viewed this way [J] balances out the disappointed hopes near the end of **5** (for ω-models and admissibles) since it exceeds them in unexpected ways; for example, Shelah used [J] to settle Whitehead's problem for L.

In the 70s, [J] was also used for new refinements of g.r.t. and for old topics in o.r.t., in particular, in β-recursion theory, and for determining the degrees of various elementary theories about objects of o.r.t. (as in Note 3). The former pokes around so promiscuously between the ω_1^X that it has brought back the old phobias about grubbiness. In view of the reservations at the end of Note 3, which do not apply to [J], some of the applications to elementary theories may constitute philosophical regress in the sense of **3**. If so, this would have counterparts in g.r.t. of the 60s. Some of them are mentioned in the next section.

7. G.r.t.: some reminiscences and object lessons. First of all, the work of Sacks et al. was accompanied by several aberrations that attracted, at the time, more attention than the simple parallel in **6**. One of them was the business of various definitions being equivalent in o.r.t., but not in g.r.t., finiteness being a case in point. It all came uncomfortably close to the differences between Cantor's infinite cardinals and ordinals, now seen to be banal (and so all the more sensational, if forgotten). Another aberration was the hope, based mainly on the few results about maximal sets alluded to in **6**, that the work on degrees in g.r.t. would lead to a mathematical proof analysis of advanced o.r.t. This hope is not reasonable in the light of the distinctions in **1**, and its failure backs them up. (By lack of time or, perhaps, the grace of God, such excesses as [Si] were avoided, with its attempt to use infinite ordinal arithmetic for Fermat's conjecture.)

Another aberration seems to be the massive literature on g.r.t. of higher types, that is, iterations of 'fat' or 'thin' power set and function space operations, as in: sets-of-sets-of... or functions-of-functions-of; at least, when viewed as follows. True, that literature reflects the central place that higher types have acquired in several branches of contemporary logic; cf. Note 8. But most of its results would mean very little to most readers of this Bulletin, and the rest would not be very compelling. The attention to higher types simply does not fit in with general experience in mathematics.

More specifically, the fact that objects of higher type occur throughout mathematics, is by itself not enough to inspire confidence in any general theory; cf. the case of ω-models in **5**. Besides, in contrast to logic (close to the foundational tradition), mathematics is accustomed to 'spreading out' in other ways, too—for example, to higher dimensions. In short, errors and omissions in current mathematical practice excepted, higher types are not (often) particularly useful scientific tools.

If so, the discovery of this defect has a direct bearing on principal themes of **0**; generally, on the differences between the broad mathematical and logical traditions, and, more specifically, for assessing the relevance of the latter. To make sure that such (object) lessons are about the logical ideals themselves and not about their incompetent use, one has to check that the literature on g.r.t. of higher types respects those ideals, and is imaginative (within its tradition, as, I believe, it is). Much the same applies to the aberrations mentioned in the first paragraph of this section.

The particular logical ideals most pertinent to g.r.t. on finite types are: a *universal language* and the precise analysis of *intended meanings* (without regard to their adequacy for, and to the scientific value of, their intended purposes); cf. Note 9 for some illustrations involving thinnish and fat hierarchies resp. The comments are brief, in line with the last paragraph of **0**; such matters are too general, and therefore too simple to be pursued efficiently in the narrow context of g.r.t.; cf. **[Kr]**.

8. The book under review. As mentioned at the outset it concerns the infinitistic part of g.r.t. with the general flavor of **5**. The main notions are not presented in the form most widely used in the successful literature. So the book is probably not an efficient introduction. Anyhow, there are plenty; for example, in **[B2]** with references to more detailed expositions.

But for the general reader the book is a treasure, albeit not in the way intended by its author. It teems with lively reminders of actually wide-spread, but repressed ideas, not merely laid out for autopsy, but expressed with conviction. At one extreme, of generality, there is blithe talk about 'natural' notions, without any hesitation over the extent to which this matter is—and occasionally is not—sensitive to background knowledge (nor over the obvious parallel from botany where perfectly natural and often pretty mushrooms can be addictive or poisonous in other ways). At another extreme, concerning the technicality of so-called inductive definitions, the author's presentation relates the current interest in them to a pun much more clearly than elsewhere in the literature known to this reviewer; cf. Note 10. In short, the book provides particularly simple object lessons of the kind adumbrated at the end of the last section.

It also serves as a reminder of striking foundational progress in this century, by contrast. In particular, on p. 256 there is a new description of o.r.t. in terms of the (perfect?) information flow among middle management, incidentally, a current target for office automation. The author talks of—quite abstract—rules for handling in-boxes and out-boxes as the crux of the matter. All this without translating any result of o.r.t. into the new lingo, let alone seeing what it might do for middle-management, on the model in **2** of what von Neumann did for data processing with the description of o.r.t. in terms of the perfect computer. By comparison, p. 256 is just a grunt; but quite typical of the grunts (and barks) that filled the popular foundational literature in its heyday in the 20s and, incidentally, a good deal of the current variety inspired by it. This becomes a reminder of progress, at least, for readers prepared to look at the contemporary informed literature on digital and analog computers and their

idealizations. By 2 and Note 2 its critical part may bite; but for good or ill, it is not reduced to grunts and barks.

NOTES

1.(a) Reminders. (i) In modern terminology, **[G1]** uses a privileged notation, say, Δ_n for $n \in \omega$, and formulas F, with a single free variable, that are invariant over ω; that is, for all models M of the system S considered, and for each $n \in \omega$: *either* $F(\Delta_n)$ is true in all M *or* $F(\Delta_n)$ is false in all M. By completeness, this is equivalent to: either $F(\Delta_n)$ is derivable (in S) *or* $\neg F(\Delta_n)$ is so derivable. Thus predicate calculus provides formal rules for o.r.t., which can be (and were) pruned to an equation calculus. (ii) Relativized o.r.t. is used to sharpen qualitative model-theoretic facts like the completeness theorem: if a countable set of axioms is r.e. in X ($\subset \omega$) so is its set of consequences; with an obvious extension to uncountable languages. (b) For more delicate results on invariant definitions—and, later, for avoiding anomalies in g.r.t.—the word 'privileged' was replaced in mathematical terms. The relevant subject of core structures is alive. Specifically, by imaginative use of o.r.t. Wilkie recently described a large class of systems that extend a weak subsystem of Peano arithmetic and have nonstandard core structures; cf. pp. 311–314 of **[DLS]**.

2.(a) The logical flavor of complexity theory comes from its choice of problems. Like o.r.t. it considers usually all formulas of some first-order theory (at best, restricted by some bound on alternating quantifiers), and classifies them by purely external parameters such as the number of symbols. The following examples of an alternative strategy come from the subjects of real closed fields, and so-called Presburger arithmetic or, more accurately, suitable abelian groups. In both cases the criterion for selecting a class of problems is the domain of efficiency of some (promising) algorithm; thus in contrast to the logical choices the virtues of these classes are not immediately apparent, but had to be discovered. (i) In **[Sm]** Newton's method is used to select polynomials with roots in certain geometric constellations; they occur, roughly speaking, with stable solutions, but not in catastrophes. To be precise, Smale's own formulation is different, using probabilistic, not geometric, language. The formulation above comes from inspecting the proof. (ii) **[BV]** selects its classes of linear diophantine equations in terms of the geometry of numbers. Its upper bounds are an order of magnitude better than known lower bounds for the general case. The possibility of such an improvement, by restriction, is of course trivial in the abstract, but not here since **[BV]** covers equations that occur in prominent problems.

(b) The current fad of talking about 'large numbers', garnished with references to the age of the universe or the number of electrons in it, has an obvious basis; not least, in the preoccupation with primitive recursion (or even Gregorczyk's classes beyond the first couple of levels). But the talk neglects both (i) familiar truisms, and (ii) experience in Higher Mathematics. Thus (i) numbers are well known to be large 'according to circumstances', and, with efficient notation, computations with numbers of large size are perfectly reliable (however embarrassing this may be to—the, in turn, particularly embarrassing—parts of the foundational literature). (ii) In **[BD]**, Baker's large bounds for a class of problems are applied to a suitable particular problem, actually going back to Diophantus; this is *discovered* to benefit from those 'large' bounds. Specifically, the particular bound, though $> 3^{10^{487}}$, is small enough for a complete solution, when supplemented by relatively little additional analysis; naturally, of aspects somewhat specific to the particular problem.

(c) At the other extreme to the revolutionary claims of complexity theory mentioned near the end of **2**, various *formal parallels* have been proposed between the separations P/NP and recursive/r.e. with, at least so far, superficial results. These do not involve any details of o.r.t., but merely the difference between deciding a question and verifying given evidence for an answer; for example, a numerical computation of $\mu(n) = m$ constitutes evidence for $m \in M$ if M is enumerated by μ, but the negation needs a general argument.

3. Here 'analog computer' means any physical system, tacitly, together with a theory according to which the initial values are under experimental control; not only the differential analyzer that originally competed with digital computers. For some (not so) fine points, for example, on the notion of initial value, cf. the review of **[PR]**. In terms of X-relativized o.r.t., an analog computer, thought of as an accessory to a digital computer, *selects* certain X, namely, those nonrecursive

outputs that arise from recursive inputs. The literature on this and other selections of privileged X is quite uneven. At one extreme, there has always been a tendency to evasion by good-humored talk of 'oracles', with the understanding that none is perfect in the sense of all others being (recursively) reducible to it. At another extreme, much attention has been given to oracles of restricted quantifier complexity Σ_n^0 or Π_n^0, among which there are perfect, also called 'complete', specimens. More recently, oracles X have been classified in terms of elementary equivalence w.r.t. fragments of such (elementary) languages as those of orders (by degrees) or lattices of sets that are r.e. in X. However, the impression of greater generality compared to, say, Higman's theorem is deceptive; for though elementary languages are indeed defined for arbitrary structures, their use—here: for classifying oracles—depends on the particular structure chosen.

4. On p. 526 in vol. 1 of A. Weil's *Collected Works*, advances in recursion theory, presumably in contrast to number theory, are expected to establish that solubility, over the *rationals*, of binary equations is undecidable. Reminder: such undecidability, over **Q**, is not even known for the class of all diophantine equations (with rational coefficients). Weil's conjecture fits the modern interest in the *solution set* of binary equations (without parameters), for example, its group-theoretic properties even when it is open whether or not the set is empty. At the same time the conjecture confirms the current gut appeal of formulations in terms of the perfect computer, an appeal noted repeatedly in this review. It remains to be seen whether specialists will come to look at proofs of undecidability also, or even primarily, for additional information on the solution set of particular equations (as in speculations about Matyasevic's original refutation of Hilbert's tenth problem and Pell's equation, say, for p-adics).

5. Here are the results alluded to in §4. (a) The incomparability result considered by Fitting (pp. 47–52) is a corollary of a general result about partial orders $<$ of subsets of ω. *Suppose each* $X \subset \omega$ *has at most countably many predecessors, and the continuum hypothesis* (*CH*) *is false. Then there are incomparable elements in* $<$, that is $\exists X \exists Y\ (\neg X < Y \,\&\, \neg Y < X)$. This is evident. If, further, the logical complexity of $<$ is suitably restricted—for example, to Π_2^1—then the conclusion holds outright by familiar conservation results extracted from (current) relative consistency proofs for $\neg$CH. Almost prehistoric 'basis' results sharpen this when $< \in \Sigma_1^0 \cup \Pi_1^0$, which certainly applies to the popular reducibilities of o.r.t. In this case there are incomparable elements of degree below $\mathbf{0}'$. (b) Concerning additional information supplied by priority arguments (and, certainly, by proofs of convergence of Hilbert's ε-substitution method compared to mere existence of some substitution), the case of Martin's proof of Borel determinacy suggests a pattern. In [M] Martin himself states only the *existence* of a winning strategy for Borel games. Others—in line with the 'neglected question' on p. 164 of [GY]—saw in [M] straightaway the quantitative side of the particular strategy supplied by Martin's style of priority argument. This shift of emphasis has become more popular since he discovered a simpler proof of the result stated in [M]. (c) *Within* o.r.t. uses of priority arguments have also been analysed in game-theoretic terms [Y]. But more often they are associated with reducing the logical complexity of—all or some of—the sets involved in problems about Turing degrees. (i) Friedberg and Mučnik sharpen (a) above so that both X and Y are r.e. (ii) Spector's result $\exists X$ (Min X), asserting the existence of a minimal, nonrecursive degree, is not suitable here since no such minimal degree is r.e. But in the variant $\exists X \exists Y$ (Min $X \wedge X \leqslant Y$), Y can be made r.e. to yield the degrees in the title of [Y]. For contrast, (d) some odd corners of o.r.t. have begun to benefit from mathematical proof analysis in terms of (Ershov's) g.r.t. on suitable function spaces of logical type in Note 9; for example, the theorem of Rice-Shapiro was interpreted as a corollary to a general fact about two topologies being equal; cf. 2.3 in [3] of [LM] (but also more algebraic styles of analysis, for example, of Myhill's theorem on creative sets in terms of permutation groups).

6. *Application* means the ternary relation: y is the value of (the function) f applied to x. The literature on the subject illustrates very effectively the conflict, broached in §0, between specifically logical and more modern mathematical traditions. (i) The former cultivates categorical axioms (like Peano's and Dedekind's, of blessed memory) as logically precise characterizations of familiar notions (of the natural, resp. real numbers). For o.r.t. this is done on pp. 121–128 of [GY] in the language of applicative structures when restricted to ω-models. (ii) Mathematical proof analysis, as explained in **1**, tends to avoid categorical axioms, thereby pin-pointing more sharply, that is, by more general axioms, abstract properties sufficient for some theorem or proof that was possibly meant originally for a specific structure.

7. (a) Most elementary results about countable admissible ordinals drop out of relativized o.r.t. applied to well orderings. Formally, by a theorem of Sacks, proved later more simply by Friedman and Jensen on pp. 77–79 of **[B1]**, *a countable* α *is admissible iff* $\alpha = \omega_1^X$ for some $X \subset \omega$. Naturally, the language of ordinals is more efficient in contexts where sets X and X' are 'equivalent' as soon as $\alpha = \alpha'$. (b) The original theorem of Cantor-Bendixson states that the sequence σ_F of derived sets of a closed set F is countable (in suitable spaces). The quantitative refinement meant in **5** is: If (the set of complementary rational intervals of) F is Π_1^1 in X then the length of σ_F is bounded by ω_1^X.

8. (a) The role of higher types in set theory is familiar. In particular, for the fat power set operation they provide the chief means of generating sets of higher cardinality, the latter being *the* condition for 'representing', in the sense of set-theoretic foundations, all structures. Also the highly advertised *axioms of infinity* serve to push up types. Analogs for thinnish power set operations play a role in g.r.t. (b) The language of higher types is useful, to electronic and biological automata, for unwinding proofs in suitable systems S. (i) Generalized o.r.t. provides so-called functional interpretations of formulas F in S, that is, $\exists\forall$ forms ranging over *recursive* objects of higher type; specifically, of all finite types even if S is in the language of second-order arithmetic. The algorithms define S-provably recursive functions if F is Π_2^0 and is proved in S from true Π_1^0 lemmas. For practical use S must be logically very weak. (ii) Algorithmically significant reductions can sometimes be achieved by such mathematically trivial changes as replacing familiar schemata of lower type by axioms with parameters of higher type, for example, in the case of induction. This fits the view that the algorithmic content is generally only a small part of the information contained in a proof; in accordance with p. 173 of **[Ca]** about *zahlentheoretischer Gehalt*, and in conflict with **[Bi]** on *numerical content*. NB The local use of logically weak systems considered here is in sharp contrast to the traditional literature with claims that are not an iota less pretentious than those of footnote 2.[3]

9. Here are some consequences for g.r.t. on finite types of the two logical ideas at the end of 7. (a) The holy grail of *one* universal structure is of course meant in contrast to a few structures adapted to many situations. (i) The defects of using *logical* types, generated from a ground type by: $(\sigma, \tau) \to (\sigma \to \tau)$, are familiar enough from set-theoretic foundations. For example, when **R** is defined in terms of the (totally disconnected) type $(0 \to 0) \to 0$ over ω, one asks: How often will knowledge about the latter tell us what we really want to know about **R**? (ii) The hankering after one scheme leads even **[E]** to speak of a 'fundamental' choice between total versus (hereditarily monotone) partial functions in his g.r.t. on so-called countable and effective operations of finite type. A moment's thought shows that the partial variety has smoother algebraic, in particular, enumeration properties while the other tends to make for computational efficiency, as in the case of multiplication by 0; validity for partial functions requires $n \cdot 0 = 0$ to be replaced, for example, by $0 \cdot 0 = 0$, $(n + 1) \cdot 0 = n \cdot 0$. (b) Analyzing intended meanings is meant in contrast to deriving notions from extended scientific experience; cf. the second sense of 'basic' in **1**, and Note 2a on the selection of problems. Now, most work in g.r.t. on the fat hierarchy of finite types, especially over ω, goes back to **[K2]** with its overtures on the meaning of Church's thesis for that structure. Contrary to a wide-spread superstition it is certainly possible to be *precise* about such intended meanings (and about their intended purposes! cf. the groans on pp. 178–194 of **[GY]**). But, as in **3**, there is the more demanding matter of *shifts of emphasis*. In the case of **[K2]** the best-known shift involves so-called normal functions, which are recursive (in the sense of **[K2]**) in the relations $^n\mathbf{E}$: equality between functions of type $n - 1$. For $n > 1$, $^n\mathbf{E}$ is the archetype of nonconstructive relations in the sense of ordinary mathematics, and so normal functions are certainly out of tune with the overtures in **[K2]**. For $n = 2$ those normal functions have been used to recover, if not to

[3] Classifications by derivability in certain formal systems are claimed to correspond to such delicate aspects of theorems and proofs as reliability (in Hilbert's business about finitist, that is, logic-free proofs) or depth (to be measured by ordinals as suggested by Turing, and given a slight twist by epigones of Gentzen). A current fad is reverse mathematics, in extreme form with the claim that formal equivalence in certain 'basic' systems expresses identity of ideas (comparable to the case of $0 = 0$ and Mordell's conjecture, which are equivalent in formal arithmetic). As at the end of **7**, such antics can serve as *object lessons*; for example, on the need for testing the significance of classifications; cf. **2** on complexity theory.

extend, a good deal of metarecursion theory in **5**. If more striking uses are ever found, one will, perhaps, speak of—preestablished?—harmony with **[K2]**, that is, with the material coming after the overtures. But for the time being the main question is wide open: For which (compelling) problems about which of the many objects in the fat hierarchy are the degrees defined in **[K2]** efficient? To be charitable, when compared to, say, those of Euclid's Book X in its time.

10. At least since Cantor the word 'recursion' applies also to the transfinite kind. This is associated with proof by induction, formerly called 'infinite descent', and brings to mind 'inductive definitions' (i.d.); cf. the gushing pp. 147–149 of [**GY**]. But these verbal associations leave open to what extent the word fits the material which has come to constitute contemporary o.r.t. More specifically, the current 'general' theory of i.d. has been handicapped by the kind of coarse classifications familiar from Note 2a. Thus, if nothing but the logical form of the defining clause of an i.d. is used, the results reduce to mere bounds on the closure ordinal, as in Cantor's definition for the perfect kernel of a closed set; cf. Note 7b. In contrast, ordinary mathematics and more imaginative logic stress the choice between i.d. with the same closure ordinals; for example, between different sets of generators for the same algebraic structure, resp. between rules of proof, a particular kind of i.d., with and without cut; cf. the passage in **2** on insignificant classifications. (*Reminder* for specialists on weak subsystems, mentioned in Note 8b(ii). Formal theories of i.d. are proof-theoretically more efficient than Dedekind's alternative, which derives them from comprehension axioms; key words: $\Pi_1^1 - CA$ and $\Pi_\infty^1 - CA$.)

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