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STRICTLY ERGODIC MODELS FOR DYNAMICAL SYSTEMS

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The action of a group G by homeomorphisms of a compact metric space X is said to be *strictly ergodic* if there is a unique Borel probability measure μ fixed by the action, and $\mu(U) > 0$ for every nonempty open set $U \subset X$. For commutative groups G (as well as for general amenable groups) this implies that the action is minimal, since if $X_0 \subsetneqq X$ is closed and G-invariant there would exist a G-invariant measure supported by X_0 which would necessarily be different from μ. Analogously one sees that the dynamical system (X, G, μ) must be ergodic. A remarkable result due to R. Jewett [**Je**] and W. Kreiger [**K**] says that for $G = \mathbf{Z}$, any ergodic action is isomorphic to a strictly ergodic system. This was extended to $G = \mathbf{R}$ by K. Jacobs [**Ja**] and M. Denker and E. Eberlein [**DE**]. Thus the topological property of strict ergodicity places no restriction on the measure theoretic properties beyond the obvious ergodicity. It is natural to ask what happens for more general groups G, and what happens, even in the case of $\mathbf{Z}$, when we look at diagrams in the category of ergodic $\mathbf{Z}$-actions rather than simply the objects themselves. In brief our results are:

(1) When G is commutative every ergodic action has a strictly ergodic model.

(2) Any diagram in the category of ergodic $\mathbf{Z}$-actions with the structure of an inverted tree, i.e., no portion of it looks like

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & & Z \end{array}$$

has a strictly ergodic model (as a diagram). However, not every measure theoretic triple

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & & Z \end{array}$$

can have a strictly ergodic model.

As a consequence of (2), we can, for example, take any ergodic $\mathbf{Z}$-action that has some point spectrum and provide for it a strictly ergodic model in which all the eigenfunctions are continuous. One can combine (1) and (2) which was formulated for $\mathbf{Z}$-actions for those interested in the classical situation.

The method of proof that was developed for (1) is flexible enough to admit further refinements. For example, suppose that $G = \mathbf{Z}^2$, and the action is given by a pair of commuting transformations T, S which are known to

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be *separately* ergodic. Then we can find a strictly ergodic model $(X;\tau,\sigma)$ where separately $(X,\tau),(X,\sigma)$ are strictly ergodic as $\mathbf{Z}$-actions. Looking to more general G's, the proof can be adapted to handle the so-called elementary amenable groups (cf. C. Chou [C]), but it becomes necessary to assume, in addition to the ergodicity, the freeness of the action.[1] This is not a consequence of strict ergodicity but rather comes from the fact that the proof relies heavily on the existence of Rohlin towers for which one needs the freeness of the action.

I would like to thank the MSRI for their hospitality during the 1983-1984 Dynamics year when this work was done.

1. $\mathbf{Z}^2$-actions. In what follows I will give a sketch of the proof of the following theorem, which is the easiest case of (1) above, and already contains the main elements of the more general results.

THEOREM 1. *If $(Y,\mathcal{C},\nu;T_1,T_2)$ is an ergodic $\mathbf{Z}^2$-action with generators T_1,T_2 then there exists a strictly ergodic $\mathbf{Z}^2$-system (X,τ_1,τ_2) with unique invariant measure μ, such that (X,τ_1,τ_2,μ) is measure theoretically isomorphic to $(Y,\mathcal{C},\nu;T_1,T_2)$.*

Let's call a set $C\in\mathcal{C}$ *uniform* if

$$\lim_{n\to\infty}\left\|\frac{1}{|S_n|}\sum_{(i,j)\in S_n}1_C(T_1^iT_2^jx)-\nu(C)\right\|_\infty=0$$

where S_n is the square $\{(i,j)\colon \max(|i|,|j|)\le n\}$. Our goal is to construct an algebra of uniform sets that is (T_1,T_2)-invariant and generates $\mathcal{C}$. Standard techniques will then give an explicit strictly ergodic model. An intermediate goal is to construct some such nontrivial algebra without worrying about generating $\mathcal{C}$, and this is what we will do now. The main device will be a nested sequence of "uniform" Rohlin towers.

DEFINITIONS. (1) *A set $B\in\mathcal{C}$ is the base of an (n,M,δ)-uniform R-tower if*

(i) $B\cap T_1^iT_2^jB=\varnothing$ *all* $(i,j)\in S_n\backslash\{(0,0)\}$,

(ii) *for a.e.* $y\in Y$,

$$\frac{|\{(i,j)\in S_M\colon T_1^iT_2^jy\in B\}|\cdot|S_n|}{|S_M|}>1-\delta.$$

(2) *A sequence of uniform R-towers $(B_i;(n_i,M_i,\delta_i))$ will be said to be nested if for all $y\in B_i$, $y'\in B_{i'}$, $i<i'$ if*

$$S_{n_i}y\cap S_{n_{i'}}y'\neq\varnothing$$

then $S_{n_i}y\subset S_{n_{i'}}y'$. (Here we begin writing S_ny in lieu of $\{T_1^iT_2^jy;(i,j)\in S_n\}$.)

[1] NOTE ADDED IN PROOF. Alan Rosenthal and I have succeeded in extending the results described here to amenable groups in general, with the added assumption that the action be free.

(3) *Finally, we will call a sequence well-nested if the uniformities persist even after some sequence of modifications in which the internal tilings of $S_{n_i}y$, $y \in B_i$ by S_{n_j}'s with $j < i$ are interchanged.*

After constructing a sequence of well-nested uniform R-towers

$$B_i, (n_i, M_i, \delta_i)$$

we proceed as follows. Given a partition P, and some $\varepsilon > 0$ we will show how to construct a $\overline{P}$, such that the algebra

$$\bigcup_{n=1}^{\infty} \bigvee_{(i,j)\in S_n} T_1^i T_2^j \overline{P}$$

is uniform, and $d(P, \overline{P}) < \varepsilon$. Applying a refinement of the ergodic theorem, we find some i_1 large enough so that most of the $S_{n_{i_1}} - P$-names across B_{i_1} have a 1-block distribution very close to the global distribution of P. Change P to P_1 so that all $S_{n_{i_1}} - P_1$-names across B_{i_1} have good 1-block distributions. The fact that B_{i_1} is $(n_{i_1}, M_{i_1}, \delta_{i_1})$-uniform gives us our first uniformity for P_1. Next we apply a refinement of the ergodic theorem again to find an i_2, so that $S_{n_{i_2}} - P_1$-names across B_{i_2} have a good "2-block" distribution. Changing $S_{n_{i_2}} - P_1$-names which aren't good involves a change in B_1. This is where the fact that the sequence is *well* nested comes in. This procedure is iterated to give the required $\overline{P}$, and our intermediate goal has been achieved. The best way to finish the proof of the theorem is to use the techniques of §2 below to catch more and more of $\mathcal{C}$.

2. The relative J-K theorem. The basic result here is the following

THEOREM 2. *If (X, τ) is a strictly ergodic $\mathbf{Z}$-action, and $(Y, \mathcal{C}, \nu, T)$ is an ergodic measure theoretic extension of (X, τ), then there exists a strictly ergodic extension of $(\hat{X}, \hat{\tau})$ of (X, τ) isomorphic to the pair $Y \to X$.*

For the proof one first finds, by well-known techniques, an extension of (X, τ) that is zero-dimensional but measure theoretically isomorphic. Then one reduces the theorem to the following

THEOREM 3. *If P is a finite partition of Y so that the algebra $\mathcal{A} = \bigcup_{n=1}^{\infty} \bigvee_{-n}^{n} T^j P$ is uniform and $\mathbf{Q}$ is any finite refinement of P and $\varepsilon > 0$ is given, then there exists a $\overline{\mathbf{Q}}$, a finite refinement of P such that*

(i) *$\bigcup_{n=1}^{\infty} \bigvee_{-n}^{n} T^j \overline{\mathbf{Q}}$ is uniform,*

(ii) *$d(\mathbf{Q}, \overline{\mathbf{Q}}) < \varepsilon$.*

The key observation here is to let the fact that the algebra generated by P is uniform help us out. We will describe the first step in the process of constructing $\overline{\mathbf{Q}}$. In the usual way, the ergodic theorem says that for n large enough most $Q - n$-names have a good 1-block distribution. Form an R-tower of height n, with base $B \in \mathcal{A}$, that fills most of Y. Most of the P-pure columns have *some* good Q-refinements. For these, modify $\mathbf{Q}$ to $\mathbf{Q}_1$ so that *all* the $\mathbf{Q}$-refinements have good q-block distribution. Denote by B_1 the base of the tower consisting of these P-pure columns. Note that $B_1 \in \mathcal{A}$, and fills most of Y, so since $\mathcal{A}$ is *uniform*, for some N, and a.e. y, the $P - N$-name of y is almost all in the tower above B_1, and so for a.e. y the $Q_1 - N$-name has good 1-block distribution.

In the next step, we will look at $P \vee \{B_1, Y \backslash B_1\}$-pure columns and continue the process. At each stage the uniformity will come from the fact that the good distributions will be on a very large set in the uniform algebra $\mathcal{A}$. It is more or less routine to push this idea through and prove Theorem 3.

The fact that not all diagrams of the type

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & & Z \end{array}$$

are realizable as strictly ergodic systems is a consequence of the following elementary result.

THEOREM 4. *If (X, τ) is strictly ergodic with unique invariant measure λ and $\pi\colon (X,\tau) \to (Y,\tau)$, $\rho\colon (X,\tau) \to (Z,\tau)$ are factor maps with $\pi^{-1}(u) \cap \rho^{-1}(v) \neq \emptyset$ for any nonempty open sets $U \subset Y$, $V \subset Z$ then $(Y, \tau, \pi \circ \lambda)$ and $(Z, \tau, \rho \circ \lambda)$ are measure theoretically disjoint.*

It follows that if we take any weakly mixing process, say (Y, T), and consider

$$\begin{array}{ccc} & Y \times Y & \\ \swarrow & & \searrow \\ Y & & Y \end{array}$$

then this diagram cannot have a uniquely ergodic model, since naturally (Y, T) is not measure theoretically disjoint from itself. It is not yet clear what the precise conditions on a diagram such as

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & & Z \end{array}$$

are that guarantee the existence of a strictly ergodic model.

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