BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 13, Number 2, October 1985

UNITS AND CLASS-GROUPS IN THE ARITHMETIC THEORY OF FUNCTION FIELDS

BY DAVID GOSS

ABSTRACT. In the arithmetic theory of function fields, there are *two* distinct analogs of classical Bernoulli numbers. It has been known for a while that one analog was intimately connected with class number formulae. Due to recent progress, one can now begin to understand the arithmetic meaning of the other.

Let $k = \mathbf{F}_r(T)$, $r = p^m$, and let $A = \mathbf{F}_r[T]$. Let $\mathcal{P} \in \operatorname{Spec}(A)$ be associated to a prime of degree d. As in [**2**] (the notation of [**2**] will be followed throughout), one can associate to $\mathcal{P}$ the abelian extension $k(\mathcal{P})$ of k with Galois group isomorphic to $A/\mathcal{P}^*$. This extension is very closely analogous to $Q(\varsigma_p)/Q$; ς_p a primitive pth root of unity. We will be interested here in the "totally real" subfield $k(\mathcal{P})_+$ of $k(\mathcal{P})$ obtained as the fixed field of $\mathbf{F}_r^* \subseteq A/\mathcal{P}^*$. In particular, our results are concerned with the p-primary class group $\operatorname{Cl}(\mathcal{P})_+^{(p)}$ of $k(\mathcal{P})_+$ and with the class group $\widetilde{\operatorname{Cl}(\mathcal{P})}_+^{(p)}$ of the ring of A-integers $\mathcal{O}(\mathcal{P})_+$ of $k(\mathcal{P})_+$. One knows that the natural projection of $\operatorname{Cl}(\mathcal{P})_+^{(p)}$ to $\widetilde{\operatorname{Cl}(\mathcal{P})}_+^{(p)}$ is surjective (see [**2**, 6.1.2]).

Associated to A one has the elements $\beta(i)$, $i \in \mathbf{N}^+$, and B_i, $i \in (r-1)\mathbf{N}^+$. These elements of k are similar to classical Bernoulli numbers in that they are special values of certain zeta-functions ([**2**]; the $\beta(i)$ occur at negative integers, the B_i occur at positive integers $\equiv 0$ $(r-1)$) and *together* possess the known properties of Bernoulli numbers; e.g., the $\beta(i)$ satisfy $\mathcal{P}$-adic congruences, etc. Let $i \in (r-1)\mathbf{N}^+$ with $i < r^d - 1$ and let $\omega_{\mathcal{P}}$ be the Teichmüller character of $A/\mathcal{P}^*$. Note that by Carlitz's von Staudt result [**2**, 5.2.4] B_i is $\mathcal{P}$-integral in this range.

In [**3**] the following result is established.

THEOREM 1. $\operatorname{Cl}(\mathcal{P})_+^{(p)}(\omega_{\mathcal{P}}^i) \neq \{0\}$ *if and only if* $\mathcal{P} | \beta(r^d - 1 - i)$.

Now let $\mathcal{E}(\mathcal{P}) \subseteq \mathcal{O}(\mathcal{P})_+^*$ be the group of cyclotomic units. From the result of Galovich-Rosen [**2**, 7.2.1] one knows that $\mathcal{O}(\mathcal{P})_+^*/\mathcal{E}(\mathcal{P})$ is a finite group of order equal to the class number of $\mathcal{O}(\mathcal{P})_+$. In fact, in [**3**] the following p-adic refinement of this result is established.

THEOREM 2. *The p-groups* $(\mathcal{O}(\mathcal{P})_+^*/\mathcal{E}(\mathcal{P}))^{(p)}(\omega_{\mathcal{P}}^i)$ *and* $\widetilde{\operatorname{Cl}(\mathcal{P})}_+^{(p)}(\omega_{\mathcal{P}}^i)$ *have the same order.*

In [**4**], S. Okada has introduced a version of the classical *Kummer homomorphism* for $\mathcal{O}(\mathcal{P})_+^*$ and has established the following.

Received by the editors October 25, 1984 and, in revised form, April 30, 1985.
1980 *Mathematics Subject Classification*. Primary 12A90.

THEOREM 3. $(\mathcal{O}(\mathcal{P})_+^*/\mathcal{E}(\mathcal{P}))^{(p)}(\omega_{\mathcal{P}}^i) \neq \{0\}$ *implies that* $\mathcal{P}|B_i$.

Thus, from Theorem 2 we obtain the following result.

THEOREM 4. $\widetilde{\mathrm{Cl}(\mathcal{P})}_+^{(p)}(\omega_{\mathcal{P}}^i) \neq \{0\}$ *implies* $\mathcal{P}|B_i$.

It is most interesting that the converse to Theorem 4 is *false*. Indeed, if it were true, Theorem 1 would then imply that $\mathcal{P}$ must divide $\beta(r^d - 1 - i)$. But for $r = 3$, we find $(T^3 - T + 1)|B_{10}$ but does *not* divide $\beta(16)$.

It can also happen that $\mathcal{P}|\beta(r^d - 1 - i)$ but does not divide B_i (e.g., $r = 3$, $(T^3 - T + 1)|\beta(8)$ but does not divide B_{18}). In this case, we find by Theorem 1 that the ith component of $\mathrm{Cl}(\mathcal{P})_+^{(p)}$ is nonzero and by [**2**, 6.1.2] is concentrated on the infinite primes. Thus, it can be *canonically* and exactly described by Stickelberger elements. This result forms a version of the *Spiegelungsatz* for function fields.

The remaining case ($\mathcal{P}|B_i$ and $\mathcal{P}|\beta(r^d - 1 - i)$) has yet to be fully explored. However, there *are* examples where $\widetilde{\mathrm{Cl}(\mathcal{P})}_+^{(p)}(\omega_{\mathcal{P}}^i) \neq \{0\}$.

Finally, the $\mathcal{P}$-divisibility properties of the $\beta(i)$ and B_i are given by the following result. It is intriguing that $\beta(i)$ appears to be more divisible than B_i.

PROPOSITION 5. (a) *Let* $i \in (r-1)\mathbf{N}^+$ *with* $i < (r^d - 1)/p$. *Then,* $\mathcal{P}|B_i$ *if and only if* $\mathcal{P}|B_{pi}$.

(b) *Let* $i \in \mathbf{N}^+$. *Then* $\mathcal{P}|\beta(i)$ *if and only if* $\mathcal{P}|\beta(pi)$.

(c) *Let* $i_1, i_2 \in \mathbf{N}^+$ *with* $i_1 \equiv i_2$ $(r^d - 1)$. *Then* $\mathcal{P}|\beta(i_1)$ *if and only if* $\mathcal{P}|\beta(i_2)$.

(d) *Let* $i \in \mathbf{N}^+$ *be of the form* $p^j(r^h - 1)$. *If* $i < r^d - 1$ *then* $\mathcal{P} \nmid B_i$.

(a) and (b) follow from the fact that $\varsigma(pi) = \varsigma(i)^p$ and the properties of the Γ_i. Part (c) follows from the congruences satisfied by the $\beta(i)$. Part (d) follows from an explicit calculation of B_i for such i by Carlitz. One should note that the numbers i of the form $r^h - 1$ are precisely those that are congruent to 0 $(r-1)$ and for which one has numerical evidence linking Γ_i and $\beta(i)$ [**2**].

REFERENCES

1. L. Carlitz, *An analogue of the von Staudt-Clausen theorem*, Duke Math. J. **3** (1937), 503–517.

2. D. Goss, *The arithmetic of function fields 2: The 'cyclotomic' theory*, J. Algebra **81** (1983), 107–149.

3. D. Goss and W. Sinnott, *Class-groups of function fields*, Duke Math. J. **52** (1985), 507–516.

4. S. Okada, *Kummer's theory for function fields* (to appear).

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210-1174