RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 13, Number 2, October 1985

OPTIMAL ISOPERIMETRIC INEQUALITIES

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It is well known (and true) that among all n-dimensional closed surfaces (boundaries of bounded regions) in R^{n+1} having n-area equal to that of the standard n-sphere $\partial B^{n+1}(0,1)$, the n-sphere itself uniquely (up to translations and sets of measure 0) encloses the largest volume. An equivalent formulation of this statement is the optimal isoperimetric inequality which asserts that

$$\mathcal{L}^{n+1}(Q) \leq \gamma(n+1)[\mathcal{H}^n(\partial Q)]^{(n+1)/n}$$

for any nonempty bounded region Q in R^{n+1} with equality if and only if (up to sets of measure 0) $Q = \operatorname{int} B^{n+1}(p,r)$ for some $p \in R^{n+1}$ and $0 < r < \infty$; here $\mathcal{L}^{n+1}$ denotes $(n+1)$-dimensional Lebesgue measure, $\mathcal{H}^n$ denotes n-dimensional Hausdorff measure, $B^{n+1}(p,r) = R^{n+1} \cap \{x : |x-p| \leq r\}$, and

$$\gamma(n+1) = \mathcal{L}^{n+1}(B^{n+1}(0,1))/[\mathcal{H}^n(\partial B^{n+1}(0,1))]^{(n+1)/n}$$

is the optimal isoperimetric constant. Q need not be the only bounded region having ∂Q as boundary.

We announce several new optimal isoperimetric inequalities of geometric measure theory, proved in [**A1**], which are valid in general dimensions and codimensions.

THEOREM. *Suppose $m \in \{1, 2, \ldots, n\}$.*

(1) *Corresponding to each nonzero m-dimensional simplicial cycle $T = \sum_i r_i \Delta_i^m$ in R^{n+1} with real coefficients there is an $(m+1)$-simplicial chain $Q = \sum_j s_j \Delta_j^{m+1}$ in R^{n+1} with real coefficients such that $\partial Q = T$ and*

$$\sum_j |s_j| \mathcal{H}^{m+1}(\Delta_j^{m+1}) < \gamma(m+1) \left[\sum_i |r_i| \mathcal{H}^m(\Delta_i^m)\right] \left[\sum_i \mathcal{H}^m(\Delta_i^m)\right]^{1/m}.$$

Received by the editors April 10, 1985.
1980 *Mathematics Subject Classification*. Primary 45F20; Secondary 53A10, 45F10.
[1]This work was supported in part by grants from the National Science Foundation.

(2) *Corresponding to each m-dimensional (real) rectifiable cycle T in R^{n+1} there is an $(m+1)$-dimensional (real) rectifiable current Q in R^{n+1} with $\partial Q = T$ and*

$$M(Q) \le \gamma(m+1)M(T)S(T)^{1/m},$$

with equality if and only if for some $0 < r < \infty$ and $-\infty < s < \infty$, T equals s times the current associated with a standard m-sphere of radius r in R^{n+1}.

A (real) rectifiable current T is one which can be written in the form $t(S, \theta, \xi)$; here S is an $(\mathcal{H}^m, m)$-rectifiable and $\mathcal{H}^m$-measurable subset of R^{n+1}, $S(T) = \mathcal{H}^m(S)$—the size of T, $\theta\colon S \to R^+$ is a positive density function, $M(T) = \int_S \theta d\mathcal{H}^m$—the mass of T, $\xi\colon S \to \Lambda_m R^{n+1}$ is a simple unit m-vector valued orientation function, and $T(\phi) = \int_S < \xi, \phi > \theta d\mathcal{H}^m$ for appropriate differential m-forms ϕ. Corresponding terminology holds for Q. If $m = n$ then Q is uniquely determined by $T = \partial Q$.

In case T is an integral current (i.e. θ is positive integer valued) *then one can require Q also to be an integral current in the optimal isoperimetric inequality.* Since $S(T) \le M(T)$ for integral T,

$$M(Q) \le \gamma(m+1)M(\partial Q)^{(m+1)/m}$$

which includes our initial inequality as a special case.

Corresponding optimal isoperimetric inequalities also hold for T and Q being members of the flat chains modulo ν in dimensions m and $m+1$ respectively for each $\nu \in \{2, 3, 4, \ldots\}$.

Largely as a consequence of [**W1** and **W2**] we conclude, additionally,

THEOREM. *Each Lipschitz map $f\colon \partial N \to R^{n+1}$ of the boundary of a compact $(m+1)$-dimensional Riemannian manifold N is the restriction of a Lipschitz mapping $g\colon N \to R^{n+1}$ such that*

$$\int_N J_{m+1} g \, d\mathcal{H}^{m+1} \le \gamma(m+1) \left(\int_{\partial N} J_m f \, d\mathcal{H}^m \right)^{(m+1)/m}$$

Here $J_{m+1}g$ denotes the $(m+1)$-dimensional Jacobian of g and $J_m f$ the m-dimensional Jacobian of f, so that the inequality dominates the Hausdorff area of g by the Hausdorff area of f.

We note that *the optimal isoperimetric constant $\gamma(m+1)$ depends only on the dimensions of the surfaces and not on their codimensions.*

The varifold estimates used in proving these optimal isoperimetric inequalities include characterizations of standard spheres (with possibly varying densities) among rectifiable varifolds. In the context of smooth manifolds these characterizations say the following. Suppose M is a compact smooth m-dimensional submanifold of R^{n+1} without boundary and that the mean curvature vectors of M do not exceed those of the standard unit m-sphere S^m in length. Then the m-area of M is not less than the m-area of S^m, with equality if and only if M is a standard unit m-sphere in R^{n+1}. More generally we have the following.

THEOREM. *Suppose $V = v(S, \theta, \tau)$ is an m-dimensional rectifiable varifold in R^{n+1} and $h: S \to R^{n+1}$ satisfies the condition*

$$\delta V(g) = \int_S g \bullet h\theta d\mathcal{H}^m$$

for each smooth vectorfield $g: R^{n+1} \to R^{n+1}$. If $\inf \theta > 0$, $\sup|h| < \infty$, and $\sup|(\tau^{\perp})_\natural h| \leq m$ (i.e. the normal components of h do not exceed m in length), then

$$S(V) = \mathcal{H}^m(S) \geq \mathcal{H}^m(\partial B^{m+1}(0,1)),$$

with equality if and only if the set S is a standard unit m-sphere in R^{n+1}.

In this theorem, $\tau: S \to G(n+1, m)$ is the unoriented tangent plane mapping of S and h is minus m times the generalized mean curvature of V.

In proving our varifold estimates we consider the boundary B of an s neighborhood of the convex hull A of $S = \operatorname{spt}||V||$, the Gauss mapping $G: B \to \partial B^{n+1}(0,1)$, and the nearest point retraction mapping $\eta: B \to A$. We use our first variation assumptions to estimate $J_n G$ and $J_m(\eta|\eta^{-1}S)$ and then use the coarea formula to express the area of G (which is $\mathcal{H}^n(\partial B^{n+1}(0,1))$) as an integral over S. Corresponding integrals occur with S replaced by $\partial B^{m+1}(0,1) \times \{0\}$ and comparison for small s shows $\mathcal{H}^m(S) \geq \mathcal{H}^m(\partial B^{m+1}(0,1) \times \{0\})$.

Proofs of optimal isoperimetric inequalities are reduced to varifold estimates essentially by utilization of compactness theorems of geometric measure theory to realize a particular $T = t(S, \theta, \xi)$ and mass minimizing Q with $M(Q) = \mathcal{L}^{m+1}(B^{m+1}(0,1))$ such that the isoperimetric inequality is an equality with the optimal constant, i.e. $M(T) \cdot S(T)^{1/m}/M(Q)$ is minimized.

Finally we establish a regularity criterion.

THEOREM. *Suppose $T = t(S, \theta, \xi)$ is an m-dimensional (real) rectifiable current in R^{n+1} and, for some $0 < \epsilon < \infty$, an associated varifold*

$$v(S, \theta + \epsilon, \tau)$$

has bounded mean curvatures. Then there is an open subset W of R^{n+1} such that $\operatorname{spt} T \cap W$ is an m-dimensional Hölder continuously differentiable submanifold of R^{n+1} and $\mathcal{H}^m(\operatorname{spt} T \sim W) = 0$.

The novel hypotheses of this theorem are satisfied by T realizing a minimum for expressions of the form $M(T) \cdot S(T)^{1/m}/M(Q)$ considered above. The hypotheses are also satisfied by T minimizing $M(T) + \epsilon S(T)$ among currents having the same boundary.

The mathematical study of various forms of isoperimetric inequalities has a substantial history; see, for example, the survey article [**OR**]. The first least-volume isoperimetric inequalities in general dimensions and codimensions of which I know appeared in [**FF**, Corollary 6.3] (with nonoptimal constants depending on dimension and codimension). The size function (necessary for real multiplicity currents) is first used in [**A2**]. For $m = n$ optimal constants for both the real and the integral case follow from [**FH**, 4.5.9(31)], while for $m = 1$ the optimal constant for an integral case is determined in [**FH**, 4.5.14].

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