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HOMOTOPY GROUPS OF THE COMPLEMENTS TO SINGULAR HYPERSURFACES

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In the early thirties O. Zariski (cf. [6, Chapter 8]) discovered a relationship between the fundamental group of the complement to a plane curve in $\mathbf{CP}^2$ and the first Betti number of cyclic covers of $\mathbf{CP}^2$ branched over this curve. At the same time, he determined the precise relationship between this Betti number and the position of singularities of the branching locus. Here we shall describe the relationship between certain higher homotopy groups of the complement to a hypersurface in $\mathbf{CP}^{n+1}$ and Hodge numbers of cyclic covers of $\mathbf{CP}^{n+1}$ branched over this hypersurface, as well as their relation to the position of singularities of the branching locus.

Let V_d^n be a hypersurface of degree d in $\mathbf{CP}^{n+1}$. Let k denote the dimension of the singular locus of V_d^n. We shall fix a generic hyperplane H (i.e., transversal to the strata of the singular locus of V_d^n). Let H_{n-k} denote a generic linear subspace of $\mathbf{CP}^{n+1}$ of dimension $n-k$. It follows from Zariski's theorem ([2]) and the computation of homotopy groups of the complement to a nonsingular hypersurface (cf. [1]) that

$$(1)\quad \pi_i(\mathbf{CP}^{n+1}-V_d^n)=\pi_i(H_{n-k}-V_d^n)=\begin{cases}\mathbf{Z}/d & i=1,\\ 0 & i=2,\dots,n-k-1,\end{cases}$$

and

$$(2)\qquad \pi_i(\mathbf{CP}^{n+1}-(V_d^n\cup H))=\begin{cases}\mathbf{Z} & i=1,\\ 0 & i=2,\dots,n-k-1,\end{cases}$$

Therefore the first homotopy group of $\mathbf{CP}^{n+1}-V_d^n$ (resp. $\mathbf{CP}^{n+1}-(V_d^n\cup H)$) which can be affected by singularities of V_d^n is π_{n-k}. From now on we shall consider $\pi_{n-k}(\mathbf{CP}^{n+1}-V_d^n)$ (resp. $\pi_{n-k}(\mathbf{CP}^{n+1}-(V_d^n\cup H))$) as $\mathbf{Z}/d$-module (resp. as $\mathbf{Z}$-module), where the action is the usual action of fundamental group on a homotopy group.

PROPOSITION 1. *Let $d\mathbf{Z}$ denote the subgroup of index d in $\mathbf{Z}$. Then $\pi_{n-k}(\mathbf{CP}^{n+1}-V_d^n)$ is isomorphic to the submodule of invariants*

$$\pi_{n-k}\left(\mathbf{CP}^{n+1}-(V_d^n\cup H)^{\operatorname{Inv} d\mathbf{Z}}\right)$$

as $\mathbf{Z}/d$-module.

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Let H_s, $s \in \mathbf{CP}^1$, be a generic pencil of hyperplanes in $\mathbf{CP}^{n+1}$, and let S be a subset in $\mathbf{CP}^1$ corresponding to nongeneric elements in H_s. Then the homotopy type of $H_s - (V_d^n \cup H)$ is independent of s for $s \notin S$. Let e_i, $i = 1,\dots,|S|$ be a system of loops in $\mathbf{CP}^1 - S$ forming a basis of $\pi_1(\mathbf{CP}^1 - S)$. The monodromy along e_i defines the automorphism E_i of $\pi_{n-k}(H_s - V_d^n \cup H))$. Let $g_1,\dots,g_T$ be a system of generators of $\pi_{n-k}(H_s - (V_d^n \cup H))$ as $\mathbf{Z}[t, t^{-1}]$-module. The following is an analog of a theorem of Van Kampen [5].

THEOREM 1. *$\pi_{n-k}(\mathbf{CP}^{n+1} - (V_d^n \cup H))$ is isomorphic as a $\mathbf{Z}[t, t^{-1}]$-module to the quotient of $\pi_{n-k}(H_s - (V_d^n \cup H))$ by submodule generated by $E_i(g_j) - g_j$, $j = 1,\dots,T$, $i = 1,\dots,|S|$.*

From now on we shall assume that V_d^n is a $\mathbf{Q}$-manifold.

PROPOSITION 2. *The homotopy group $\pi_{n-k}(\mathbf{CP}^{n+1} - (V_d^n \cup H)) \otimes \mathbf{Q}$ is a torsion $\mathbf{Q}[t, t^{-1}]$-module, i.e., is isomorphic to $\oplus_{i=1}^{l}\mathbf{Q}[t, t^{-1}]/\Delta_i$, where Δ_i are certain elements of $\mathbf{Q}[t, t^{-1}]$. Moreover $\Delta_i(1) = 1$.*

We call $P = \prod_{i=1}^{l}\Delta_i$ the $\mathbf{Q}[t, t^{-1}]$-order of π_{n-k}.

Let $\tilde{X}_m$ denote a desingularization of the m-fold cyclic cover of $\mathbf{CP}^{n+1}$ branched over V_n^d and possibly the hyperplane H (i.e., a desingularization of projective closure of the affine hypersurface $z_0^m = f(z_1,\dots,z_{n+1})$, where f is an equation of V_d^n).

PROPOSITION 3. *Let $h^{n,0}(\tilde{X}_m) = \dim H^n(\tilde{X}_m, O_{\tilde{X}_m})$. Then $h^{n,0}(\tilde{X}_m)$ does not exceed the sum of the numbers of common roots of Δ_i and $t^m - 1$ for all i.*

From now on we shall assume that V_d^n has only isolated singularities, i.e., $k = 0$. Note that use of Zariski's theorem as above allows us to reduce the questions considered here to this special case. Let B_c denote a small ball about a singular point c of hypersurface V_d^n. We denote by P_c the order of $\pi_n(B_c - V_d^n) \otimes \mathbf{Q}$ as $\mathbf{Q}[\pi_1(B_c - V_d^n)] = \mathbf{Q}[t, t^{-1}]$-module. Note that P_c is in fact just the characteristic polynomial of the local monodromy operator of the singularity c. Let $T(H)$ denote a small tubular neighbourhood of H in $\mathbf{CP}^{n+1}$. Denote by P_∞ the order of $\pi_n(\partial T(H) - V_d^n) \otimes \mathbf{Q}$ as $\mathbf{Q}[t, t^{-1}]$-module, where the module structure is obtained from the natural homomorphism

$$\pi_1(\partial T - V_d^n) \to \mathbf{Z}$$

defined by linking with V_d^n.

THEOREM 2. (1) *P divides $\prod P_c$, where the product is taken over all singular points of V_d^n.*

(2) *P divides P_∞.*

EXAMPLE. Let V_d^n have only singularities of Brieskorn type, i.e. which are locally given by the equation

$$z_1^{p_1} + \cdots + z_{n+1}^{p_{n+1}} = 0. \tag{3}$$

If $p_1,\dots,p_{n+1}$ are pairwise relatively prime then V_d^n is a $\mathbf{Q}$-manifold. The order of local π_n for each singularity (3) is equal to $\prod_{0 \leqslant \alpha_i < p_i}(t - \omega_1^{\alpha_1} \cdots \omega_{n+1}^{\alpha_{n+1}})$, where ω_i is a primitive root of unity of order p_i. Moreover P_∞ is equal to

$$\prod_{0 \leqslant \alpha_i < d} (t - \omega_0^{\alpha_1} \cdots \omega_0^{\alpha_{n+1}}),$$

where ω_0 is a primitive root of unity of degree d. In particular, Proposition 2 implies that if all p_i's are relatively prime to d then

$$\pi_n(\mathbf{CP}^{n+1} - (V_d^n \cap H)) \otimes \mathbf{Q} = 0,$$

and Proposition 3 implies that if m and d are relatively prime then $h^{n,0}(\tilde{X}_m) = 0$.

THEOREM 3. *Fix $p_1, \ldots, p_{n+1}$ such that $\sum_{i=1}^{n+1} 1/p_i < 1$ and assume that all singularities of V_d^n are of type* (3). *Then $h^{n,0}(\tilde{X}_d)$ is greater than or equal to the superabundance of system of hypersurfaces of degree $d(1 - \sum_{i=1}^{n+1} 1/p_i) - n - 2$ passing through the singularities of V_d^n.*

COROLLARY. *Let $q_i = (\prod_{j=1}^{n+1} p_j)/p_i$ and let V_d^n be given by the equation $\sum_{i=1}^{n+1} f_{q_i}^{p_i} = 0$, where f_{q_i} denotes a generic form of degree q_i. Then*

$$\pi_n(\mathbf{CP}^{n+1} - (V_d^n \cup H)) \otimes Q \neq 0.$$

Indeed, it follows from the generalized theorem of Cayley-Bacharach [4] that the superabundance in Theorem 3 is equal to 1, and Proposition 3 implies nonvanishing of $\pi_n \otimes \mathbf{Q}$.

On the other hand, the following theorem gives many examples of singular hypersurfaces for which π_2 of the complement is trivial.

THEOREM 4. *Let V be a nonsingular simply connected projective manifold of dimension n and let p be a generic projection of V into $\mathbf{CP}^{n+1}$. Then*

$$\pi_2(\mathbf{CP}^{n+1} - p(V)) \otimes \mathbf{Q} = 0.$$

The proofs of these results are based on the fact that in the case $k = 0$, $\pi_n(\mathbf{CP}^{n+1} - (V_d^n \cup H))$ can be identified with the n-dimensional homology group of the infinite cyclic cover of $\mathbf{CP}^{n+1} - (V_d^n \cup H)$. It makes this π_n a high-dimensional analog of the Alexander module of plane algebraic curves (cf. [3]). Theorems 2 and 3 are analogs of their counterparts for plane algebraic curves. An additional ingredient, however, is the use of the mixed Hodge structure on the homology of finite cyclic covers of $\mathbf{CP}^{n+1} - (V_d^n \cup H)$. The main step in the proof of Theorem 4 is to show that the action of

$$\pi_1(\mathbf{CP}^{n+1} - (p(V) \cup H)) = \mathbf{Z}$$

on $\pi_2(\mathbf{CP}^{n+1} - (p(V) \cup H))$ is trivial. This allows us to identify the latter group with $H_2(\mathbf{CP}^{n+1} - p(V) \cup H, \mathbf{Q})$. Triviality of $H_2(\mathbf{CP}^{n+1} - p(V) \cup H, \mathbf{Q})$ follows from a direct homology computation.

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