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SOME EXTREMAL FUNCTIONS IN FOURIER ANALYSIS

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1. Introduction. In the late 1930s A. Beurling observed that the entire function

$$(1.1)\quad B(z)=\left(\frac{\sin \pi z}{\pi}\right)^2\left\{\sum_{n=0}^{\infty}(z-n)^{-2}-\sum_{m=-\infty}^{-1}(z-m)^{-2}+2z^{-1}\right\}$$

satisfies a simple and useful extremal property. We have

$$(1.2)\qquad \operatorname{sgn}(x)\leqslant B(x)$$

for all real x and

$$(1.3)\qquad \int_{-\infty}^{\infty} B(x)-\operatorname{sgn}(x)\,dx=1.$$

The function $B(z)$ is entire of exponential type 2π, and Beurling showed that if $F(z)$ is any entire function of exponential type 2π satisfying $\operatorname{sgn}(x)\leqslant F(x)$ for all real x, then

$$(1.4)\qquad \int_{-\infty}^{\infty} F(x)-\operatorname{sgn}(x)\,dx\geqslant 1.$$

Moreover, he showed that there is equality in (1.4) if and only if $F(z)=B(z)$. As an application Beurling found an interesting inequality for almost periodic functions (we include it here in Theorem 15), but his results were never published.

In 1974 A. Selberg used the function $B(z)$ to obtain a sharp form of the large sieve inequality. Selberg noted that if $\chi_E(x)$ is the characteristic function of the interval $E=[\alpha,\beta]$ and

$$(1.5)\qquad C_E(z)=\frac{1}{2}\{B(\beta-z)+B(z-\alpha)\},$$

then

$$(1.6)\qquad \chi_E(x)\leqslant C_E(x)$$

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for all real x. For $x \neq \alpha$ and $x \neq \beta$ we have

$$\chi_E(x) = \frac{1}{2}\{\operatorname{sgn}(\beta - x) + \operatorname{sgn}(x - \alpha)\},$$

so (1.6) follows immediately from (1.2). Since $C_E(x)$ is continuous, the restrictions on x can be removed. By using (1.3) and (1.5) Selberg observed that $C_E(x)$ is integrable along the real axis and

$$\int_{-\infty}^{\infty} C_E(x) - \chi_E(x)\, dx = 1. \tag{1.7}$$

Of course, $C_E(z)$ is entire of exponential type 2π, but now, for applications, it is usually more convenient to work with an equivalent property of the Fourier transform of C_E. Specifically, the Fourier transform

$$\hat{C}_E(t) = \int_{-\infty}^{\infty} C_E(x) e(-tx)\, dx$$

(where we write $e(u) = e^{2\pi i u}$) is a continous function supported on $[-1, 1]$.

To illustrate one of the simplest applications of Selberg's function, let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers and

$$f(x) = \sum_{n=1}^{N} a(n) e(\lambda_n x)$$

an almost periodic trigonometric polynomial. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_N$ are well spaced in the sense that $|\lambda_n - \lambda_m| \geqslant 1$ whenever $n \neq m$. Using inequality (1.6) we have

$$\begin{aligned} \int_\alpha^\beta |f(x)|^2\, dx &\leqslant \int_{-\infty}^{\infty} C_E(x) |f(x)|^2\, dx \\ &= \sum_{n=1}^{N} \sum_{m=1}^{N} a(n)\overline{a(m)} \int_{-\infty}^{\infty} C_E(x) e((\lambda_n - \lambda_m)x)\, dx \\ &= \sum_{n=1}^{N} \sum_{m=1}^{N} a(n)\overline{a(m)}\, \hat{C}_E(\lambda_m - \lambda_n). \end{aligned} \tag{1.8}$$

Now $\hat{C}_E(\lambda_m - \lambda_n) = 0$ if $|\lambda_m - \lambda_n| \geqslant 1$, so all of the nondiagonal terms on the right of (1.8) are zero. It follows that

$$\int_\alpha^\beta |f(x)|^2\, dx \leqslant \hat{C}_E(0) \sum_{n=1}^{N} |a(n)|^2 = (\beta - \alpha + 1) \sum_{n=1}^{N} |a(n)|^2.$$

More generally, if $|\lambda_n - \lambda_m| \geqslant \delta > 0$ for $n \neq m$, then an obvious change of variables in the previous argument leads to the upper bound

$$\int_\alpha^\beta |f(x)|^2\, dx \leqslant (\beta - \alpha + \delta^{-1}) \sum_{n=1}^{N} |a(n)|^2. \tag{1.9}$$

By modifying his original construction Selberg found an entire function $c_E(z)$ of exponential type 2π which satisfies the minorizing inequality $c_E(x) \leqslant \chi_E(x)$ for all real x and also

$$\int_{-\infty}^{\infty} \chi_E(x) - c_E(x)\, dx = 1.$$

Of course, this provides a lower bound which, when combined with (1.9), can be written as

$$(1.10)\qquad \int_\alpha^\beta |f(x)|^2\,dx = (\beta - \alpha + \theta\delta^{-1}) \sum_{n=1}^{N} |a(n)|^2$$

with $-1 \leqslant \theta \leqslant 1$.

The identity (1.10) was also obtained by Montgomery and Vaughan [**M-V**] from a generalization of Hilbert's inequality. In fact, their form of Hilbert's inequality can also be established directly from Beurling's inequality (1.2) and knowledge of the Fourier transform of $B(x) - \operatorname{sgn}(x)$. We provide the details in Theorem 16.

The functions $c_E(z)$ and $C_E(z)$ occur as special cases of a general method for constructing entire functions of prescribed exponential type which minorize or majorize a given function of bounded variation. We describe this result in §4.

Let

$$S(x) = \sum_{n=M+1}^{M+N} a(n)e(nx)$$

be a trigonometric polynomial with period 1, and let $\xi_1, \xi_2, \ldots, \xi_R$ be real numbers which are well spaced modulo 1. Specifically, we suppose that $\|\xi_r - \xi_s\| \geqslant \delta > 0$ for $r \neq s$, where $\|x\|$ is the distance from x to the nearest integer. In its most basic setting the large sieve is an inequality of the form

$$(1.11)\qquad \sum_{r=1}^{R} |S(\xi_r)|^2 \leqslant \Delta(N, \delta) \sum_{n=M+1}^{M+N} |a(n)|^2.$$

By using the function $C_E(z)$ Selberg established (1.11) with $\Delta(N, \delta) = N - 1 + \delta^{-1}$, which is sharp. An essentially equivalent bound with $\Delta(N, \delta) = N + \delta^{-1}$ was obtained at about the same time by Montgomery and Vaughan [**Mon**]. Selberg's proof of (1.11) is simple and direct. Let $F(z)$ be the entire function of exponential type $2\pi\delta$ defined by $F(z) = C_E(\delta z)$, where $E = [\delta(M+1), \delta(M+N)]$. It follows that $F(x) \geqslant 0$ for all real x and $F(x) \geqslant 1$ for $M + 1 \leqslant x \leqslant M + N$. Since

$$\hat{F}(t) = \delta^{-1}\hat{C}_E(\delta^{-1}t),$$

we see that $\hat{F}$ is supported on $[-\delta, \delta]$ and $\hat{F}(0) = N - 1 + \delta^{-1}$. By a theorem of Fejer [**Boa**, pp. 124–126] there is an entire function $f(z)$ such that $F(x) = |f(x)|^2$ for all real x, $f(x)$ is obviously in $L^2(\mathbf{R})$, and

$$f(z) = \int_{-\delta/2}^{\delta/2} \hat{f}(t)e(zt)\,dt,$$

that is, $\hat{f}$ is supported on $[-\delta/2, \delta/2]$. Now define

$$S^*(x) = \sum_{n=M+1}^{M+N} a(n)f(n)^{-1}e(nx).$$

The identity

$$(1.12)\qquad S(x) = \int_{-\delta/2}^{\delta/2} \hat{f}(u)S^*(u + x)\,du$$

follows immediately. If we apply Cauchy's inequality to the right side of (1.12) we find that

$$(1.13)\qquad |S(\xi_r)|^2 \leq \int_{-\delta/2}^{\delta/2} |\hat{f}(u)|^2\, du \int_{-\delta/2}^{\delta/2} |S^*(u+\xi_r)|^2\, du$$
$$= \hat{F}(0)\int_{\xi_r-\delta/2}^{\xi_r+\delta/2} |S^*(u)|^2\, du.$$

Finally, we sum both sides of (1.13) over r and use the well spacing of $\xi_1, \xi_2, \ldots, \xi_R \bmod 1$. In this way we obtain

$$(1.14)\qquad \sum_{r=1}^{R} |S(\xi_r)|^2 \leq \hat{F}(0)\int_0^1 |S^*(u)|^2\, du$$
$$= (N-1+\delta^{-1}) \sum_{n=M+1}^{M+N} |a(n)|^2 F(n)^{-1}$$
$$\leq (N-1+\delta^{-1}) \sum_{n=M+1}^{M+N} |a(n)|^2.$$

This method for proving the large sieve is implicit in [**Sel**, p. 215].

In view of the extremal property satisfied by Buerling's function $B(z)$, one might expect that a similar property would hold for the function $C_E(z)$. This is indeed the case, but only if the length $\beta - \alpha$ of the interval E is an integer. Selberg has shown that if $F(z)$ is any entire function of exponential type 2π which majorizes $\chi_E(x)$ along the real axis, then

$$(1.15)\qquad \int_{-\infty}^{\infty} F(x) - \chi_E(x)\, dx \geq 1,$$

provided that $\beta - \alpha$ is an integer. In this case $C_E(z)$ is clearly extremal; however, it is not unique. The set of all extremal functions for (1.15) was determined by Selberg (see [**GV$_2$**, p. 289]). If $\beta - \alpha$ is not an integer, then inequality (1.15) is false in general. B. Logan [**Log**] has found the corresponding extremal function for $\beta - \alpha$ not an integer and established that it is unique.

Although Selberg's function $C_E(z)$ is not extremal for every interval E, it has proved to be a useful device for establishing several important inequalities. A further account of its applications in connection with the large sieve is contained in [**Mon, GV$_1$, GV$_2$** and **Sel**]. Our purpose here is to give a more general discussion of the extremal problems which motivated the construction of $B(z)$ and $C_E(z)$ and to provide some additional applications.

Our notation for Fourier transforms, Fourier series, and convolutions follows that of Stein and Weiss [**StW**]. We say that a function $f\colon \mathbf{R} \to \mathbf{C}$ is *normalized* if

$$(1.16)\qquad \lim_{h\to 0+} \frac{1}{2}\{f(x+h) + f(x-h)\} = f(x)$$

for every real x. An entire function $F(z)$, $z = x + iy$, is said to have *exponential type* $\sigma \geq 0$ if, for every $\varepsilon > 0$,

$$|f(z)| \leq A(\varepsilon)e^{(\sigma+\varepsilon)|z|}$$

for all z and some positive constant $A(\varepsilon)$ which may depend on ε. We write $\|x\|$ for the distance from the real number x to the nearest integer.

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2. Special functions. We have noted that the function $B(x)$ majorizes $\operatorname{sgn}(x)$ and minimizes the integral on the left of (1.8). Before proving these facts about $B(x)$, we consider the following simpler question: How can $\operatorname{sgn}(x)$ be *approximated* by an entire function $F(z)$ of exponential type σ in such a way that the integral

$$\int_{-\infty}^{\infty} |F(x) - \operatorname{sgn}(x)|dx \tag{2.1}$$

is minimized? This problem can be reformulated in terms of Beurling's theory of minimal extrapolation [**Beu**], and a solution can be constructed from a general method of Sz. Nagy [**SNa**] (see also Shapiro [**Sha**, Chapter 7]). Here, however, we shall take a more direct approach which can be suitably modified to deal with $B(x)$ and the problem of majorizing $\operatorname{sgn}(x)$.

If $F(z)$ is an entire function of exponential type π and $F(x)$ is bounded on $\mathbf{R}$, then $F(z)$ can be represented by the interpolation formula (Timan [**Tim**, p. 183] or Zygmund [**Zyg**, vol. II, p. 275])

$$\begin{aligned} F(z) = {} & \left(\frac{\sin \pi z}{\pi}\right) \\ & \cdot \left\{ F(0)z^{-1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n F(n)\left((z-n)^{-1} + n^{-1}\right) + F'(0)\right\}. \end{aligned} \tag{2.2}$$

This suggests that the special function

$$G(z) = \left(\frac{\sin \pi z}{\pi}\right)\left\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \operatorname{sgn}(n)\left((z-n)^{-1} + n^{-1}\right) + \log 4\right\}, \tag{2.3}$$

which interpolates $\operatorname{sgn}(x)$ at the integers, should be a good approximation to $\operatorname{sgn}(x)$ on $\mathbf{R}$. In fact, $G(z)$ is the unique entire function which minimizes the integral (2.1) with $\sigma = \pi$. It will be clear from the following lemmas that our choice of $G'(0) = \log 4$ is the right one.

LEMMA 1. *The function $G(x)$ satisfies*

$$0 \leqslant \operatorname{sgn}(\sin \pi x)\{\operatorname{sgn}(x) - G(x)\} \leqslant \left|\frac{\sin \pi x}{\pi x}\right|(1 + |x|)^{-1} \tag{2.4}$$

for all real x.

PROOF. We define functions $U_N(z)$ and $G_N(z)$ by

$$U_N(z) = \left(\frac{\sin \pi z}{\pi}\right)\left\{ z^{-1} + \sum_{\substack{n=-N \\ n\neq 0}}^{N} (-1)^n\left((z-n)^{-1} + n^{-1}\right)\right\} \tag{2.5}$$

and

$$(2.6)\quad G_N(z) = \left(\frac{\sin \pi z}{\pi}\right)\left\{\sum_{\substack{n=-N\\ n\neq 0}}^{N} (-1)^n \operatorname{sgn}(n)\left((z-n)^{-1} + n^{-1}\right) - \sum_{\substack{m=-N\\ m\neq 0}}^{N} (-1)^m \operatorname{sgn}(m) m^{-1}\right\}.$$

It follows that $U_N(z) \to 1$ and $G_N(z) \to G(z)$ uniformly on compact subsets of $\mathbf{C}$ as $N \to \infty$. From (2.5) and (2.6) we have

$$(2.7)\quad \operatorname{sgn}(\sin \pi x)\{U_{2N}(x) - G_{2N}(x)\} = \left|\frac{\sin \pi x}{\pi}\right|\left\{x^{-1} + 2\sum_{n=1}^{2N} (-1)^n (x+n)^{-1}\right\}.$$

Next we assume that $x > 0$ and reorganize the sum on the right of (2.7). We find that

$$\begin{aligned} x^{-1} + 2\sum_{n=1}^{2N} (-1)^n (x+n)^{-1} &= \sum_{n=0}^{2N} (-1)^n (x+n)^{-1} + \sum_{m=1}^{2N} (-1)^m (x+m)^{-1} \\ &= \sum_{l=0}^{N-1} \left\{(x+2l)^{-1}(x+2l+1)^{-1} - (x+2l+1)^{-1}(x+2l+2)^{-1}\right\} \\ &\quad + (x+2N)^{-1} \\ &= \sum_{n=0}^{2N-1} (-1)^n (x+n)^{-1}(x+n+1)^{-1} + (x+2N)^{-1}. \end{aligned}$$

Letting $N \to \infty$ we have

$$(2.8)\quad \operatorname{sgn}(\sin \pi x)\{1 - G(x)\} = \left|\frac{\sin \pi x}{\pi}\right|\left\{\sum_{n=0}^{\infty} (-1)^n (x+n)^{-1}(x+n+1)^{-1}\right\}.$$

Finally, we use the estimate

$$0 \leqslant \sum_{n=0}^{\infty} (-1)^n (x+n)^{-1}(x+n+1)^{-1} \leqslant x^{-1}(x+1)^{-1}$$

for the alternating series in (2.8) to deduce that (2.4) holds for $x > 0$. Since the expressions in (2.4) are even functions, the inequality must also hold for $x < 0$. The case $x = 0$ is trivial, so the lemma is proved.

LEMMA 2. *The function* $I(z) = \frac{1}{2}G'(z)$ *satisfies*

$$(2.9)\quad I(x) \ll (1+x^2)^{-1}$$

for all real x. *Thus* $I(x)$ *is integrable, and its Fourier transform is given by*

$$(2.10)\quad \hat{I}(t) = \begin{cases} \pi t \cot \pi t & \text{if } |t| \leqslant 1/2, \\ 0 & \text{if } |t| \geqslant 1/2. \end{cases}$$

PROOF. We write the function $G_N(z)$, defined by (2.6), in the form

$$G_N(z) = \sum_{n=-N}^{N} \operatorname{sgn}(n) \frac{\sin \pi(z-n)}{\pi(z-n)} = \sum_{n=-N}^{N} \operatorname{sgn}(n) \int_{-1/2}^{1/2} e((z-n)t)\,dt. \tag{2.11}$$

By applying $\frac{1}{2}(d/dz)$ to both sides of (2.11) and using the identity

$$\sum_{n=-N}^{N} \operatorname{sgn}(n) e(-nt) = -i \cot \pi t + i \left\{ \frac{\cos \pi(2N+1)t}{\sin \pi t} \right\}, \tag{2.12}$$

we find that

$$\frac{1}{2} G_N'(z) = \int_{-1/2}^{1/2} \{\pi t \cot \pi t\} e(tz)\,dt - \int_{-1/2}^{1/2} \left\{ \frac{\pi t}{\sin \pi t} \right\} (\cos \pi(2N+1)t) e(tz)\,dt. \tag{2.13}$$

As $N \to \infty$, the second integral on the right of (2.13) converges to zero by the Riemann-Lebesgue lemma. This establishes the representation

$$I(z) = \int_{-1/2}^{1/2} \{\pi t \cot \pi t\} e(tz)\,dt. \tag{2.14}$$

Next we define $\psi(t) = \pi t \cot \pi t$ for $-1 < t < 1$ and integrate by parts twice in (2.14). We obtain

$$I(z) = -(2\pi z)^{-2} \left\{ \pi^2 \cot \pi z + \int_{-1/2}^{1/2} \psi''(t) e(tz)\,dt \right\},$$

which proves the estimate (2.9). Of course, this also shows that $I(x)$ is integrable, and (2.10) then follows from (2.14) by the Fourier inversion formula.

COROLLARY 3. *The Fourier transform of the function $D(x) = G(x) - \operatorname{sgn}(x)$ is given by*

$$\hat{D}(t) = \begin{cases} 0 & \text{if } t = 0, \\ (\pi i t)^{-1}\{\hat{I}(t) - 1\} & \text{if } t \neq 0. \end{cases} \tag{2.15}$$

PROOF. Since D is odd we may assume that $t \neq 0$. Then (2.15) is obtained from the formula

$$\hat{I}(t) - 1 = \frac{1}{2} \int_{-\infty}^{\infty} e(-tx)\,dD(x)$$

after an integration by parts.

THEOREM 4. *If $F(z)$ is an entire function of exponential type σ, then*

$$\int_{-\infty}^{\infty} |F(x) - \operatorname{sgn}(x)|\,dx \geqslant \frac{\pi}{\sigma} \tag{2.16}$$

for $\sigma > 0$, *and*

$$\int_{-\infty}^{\infty} |F(x) - \operatorname{sgn}(x)|\, dx = +\infty$$

for $\sigma = 0$. *Moreover, there is equality in* (2.16) *if and only if* $F(z) = G(\sigma\pi^{-1}z)$.

PROOF. We begin by assuming that $\sigma > 0$; then, by an obvious change of variables, we may assume that $\sigma = \pi$. Let $F(z)$ be an entire function of exponential type π such that

$$\int_{-\infty}^{\infty} |F(x) - \operatorname{sgn}(x)|\, dx < \infty. \tag{2.17}$$

By Lemma 1 the function $\operatorname{sgn}(x) - G(x)$ is integrable, so, by the triangle inequality, $F(x) - G(x)$ is integrable. Since $F(z) - G(z)$ has exponential type π, it follows from a classical result of Polya and Plancheral **[P-P]** that $F'(x) - G'(x)$ is integrable. Finally, $G'(x)$ is integrable by Lemma 2, and thus $F'(x)$ must be integrable. This also shows that $F(x)$ and $F(x) - \operatorname{sgn}(x)$ have bounded variation on **R**.

For the remainder of the proof we write $\psi(x) = F(x) - \operatorname{sgn}(x)$ and $\varphi(x) = \frac{1}{2}F'(x)$. The Fourier transforms of ψ and φ are related by the identity

$$\hat{\psi}(t) = (\pi i t)^{-1}\{\hat{\varphi}(t) - 1\} \tag{2.18}$$

for $t \neq 0$. This follows immediately from

$$\hat{\varphi}(t) - 1 = \frac{1}{2}\int_{-\infty}^{\infty} e(-tx)\, d\psi(x)$$

and an integration by parts. Since $\varphi(z) = \frac{1}{2}F'(z)$ is an entire function of exponential type π, the transform $\hat{\varphi}(t)$ is continuous and supported on $[-\frac{1}{2}, \frac{1}{2}]$. Thus,

$$\hat{\psi}(t) = -(\pi i t)^{-1} \tag{2.19}$$

if $|t| \geqslant \frac{1}{2}$. Next we observe that $\operatorname{sgn}(\sin \pi x)$ has period 2 and the Fourier series expansion

$$\operatorname{sgn}(\sin \pi x) = \frac{2}{\pi i}\sum_{k=-\infty}^{\infty} (2k+1)^{-1} e\big((k + \tfrac{1}{2})x\big). \tag{2.20}$$

As $\operatorname{sgn}(\sin \pi x)$ is a normalized function of bounded variation on $[0, 2]$, this Fourier expansion converges at every point x and the partial sums are uniformly bounded. Using (2.19) and (2.20) we obtain the lower bound

$$\begin{aligned}\int_{-\infty}^{\infty} |F(x) - \operatorname{sgn}(x)|\, dx &\geqslant \left|\int_{-\infty}^{\infty} \psi(x)\operatorname{sgn}(\sin \pi x)\, dx\right| \\ &= \left|\frac{2}{\pi i}\sum_{k=-\infty}^{\infty} (2k+1)^{-1}\int_{-\infty}^{\infty} \psi(x) e\big((k+\tfrac{1}{2})x\big)\, dx\right| \\ &= \left|\frac{2}{\pi i}\sum_{k=-\infty}^{\infty} (2k+1)^{-1}\hat{\psi}\big(-(k+\tfrac{1}{2})\big)\right| \\ &= \frac{4}{\pi^2}\sum_{k=-\infty}^{\infty} (2k+1)^{-2} = 1.\end{aligned} \tag{2.21}$$

It is clear from the lower bound in Lemma 1 that there is equality in (2.21) if $F(z) = G(z)$. On the other hand, if we assume that there is equality in (2.21) then $\{F(x) - \operatorname{sgn}(x)\}\operatorname{sgn}(\sin \pi x)$ does not change sign. Since $F(x)$ is continuous, we easily deduce that $F(n) = \operatorname{sgn}(n)$ at each integer n. From the interpolation formula (2.2) and (2.3) it follows that

$$F(z) = G(z) + \beta \sin \pi z$$

for some constant β. But we have already seen that $F(x) - G(x)$ is integrable; thus $\beta = 0$.

If $F(z)$ has exponential type zero, then it is of exponential type σ for every $\sigma > 0$. Therefore (2.16) holds for every $\sigma > 0$; hence (2.17) is false.

We now turn our attention to the problem of majorizing $\operatorname{sgn}(x)$ by entire functions of exponential type. If $F(z)$ is entire of exponential type 2π, bounded on $\mathbf{R}$, and an *odd function*, then F can be represented by the interpolation formula

$$(2.22) \quad F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \cdot \left\{ \sum_{m=-\infty}^{\infty} F(m)(z-m)^{-2} + \lim_{N \to \infty} \sum_{n=-N}^{N} F'(n)(z-n)^{-1} \right\}.$$

This is a special case of a more general identity which we prove in §3. Of course, the advantage of (2.22) over (2.2) is that (2.22) interpolates both F and F' at the integers. The price we pay for this is an increase in the exponential type from π to 2π. In view of (2.22) we define three special functions, each having exponential type 2π, as follows:

$$H(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-\infty}^{\infty} \operatorname{sgn}(m)(z-m)^{-2} + 2z^{-1} \right\},$$

$$J(z) = \tfrac{1}{2}H'(z), \quad \text{and} \quad K(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2.$$

We note that $H(z) + K(z)$ is the function $B(z)$ defined by (1.1).

LEMMA 5. *For all real x we have*

$$(2.23) \qquad |H(x)| \leqslant 1$$

and

$$(2.24) \qquad |\operatorname{sgn}(x) - H(x)| \leqslant K(x).$$

PROOF. It suffices to show that

$$(2.25) \qquad 1 - K(x) \leqslant H(x) \leqslant 1$$

holds for $x > 0$. The result will then follow easily from the observation that $\operatorname{sgn}(x)$ and $H(x)$ are odd functions.

From the identity

$$\sum_{m=-\infty}^{\infty} (z-m)^{-2} = \left(\frac{\pi}{\sin \pi z}\right)^2$$

we have (with $x > 0$)

$$H(x) = 1 + \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{2x^{-1} - x^{-2} - 2\sum_{m=1}^{\infty} (x+m)^{-2}\right\}. \tag{2.26}$$

By the arithmetic-geometric mean inequality,

$$\begin{aligned} x^{-2} + 2\sum_{m=1}^{\infty} (x+m)^{-2} &= \sum_{m=0}^{\infty} \left\{(x+m)^{-2} + (x+m+1)^{-2}\right\} \\ &\geqslant 2\sum_{m=0}^{\infty} (x+m)^{-1}(x+m+1)^{-1} = 2x^{-1}. \end{aligned}$$

It follows that the inequality on the right of (2.25) holds for $x > 0$. On the other hand,

$$\sum_{m=1}^{\infty} (x+m)^{-2} \leqslant \sum_{m=0}^{\infty} (x+m)^{-1}(x+m+1)^{-1} = x^{-1},$$

which, when combined with (2.26), confirms the inequality on the left of (2.25).

The bound (2.24) obviously implies Beurling's inequality (1.2) and also shows that $H(x) - \operatorname{sgn}(x)$ is integrable. To obtain (1.3) we note that

$$\begin{aligned} \int_{-\infty}^{\infty} B(x) - \operatorname{sgn}(x)\, dx &= \int_{-\infty}^{\infty} K(x)\, dx + \int_{-\infty}^{\infty} H(x) - \operatorname{sgn}(x)\, dx \\ &= \int_{-\infty}^{\infty} K(x)\, dx = 1, \end{aligned}$$

since $H(x) - \operatorname{sgn}(x)$ is odd.

Next we consider the function $J(z)$.

THEOREM 6. *The function $J(x)$ is integrable and satisfies*

$$J(x) \ll (1 + |x|)^{-3} \tag{2.27}$$

for all real x. The Fourier transform of $J(x)$ is given by

$$\hat{J}(t) = \begin{cases} 1 & \text{if } t = 0, \\ \pi t(1 - |t|)\cot \pi t + |t| & \text{if } 0 < |t| < 1, \\ 0 & \text{if } 1 \leqslant |t|. \end{cases} \tag{2.28}$$

The function $\hat{J}(t)$ is even, nonnegative, continuously differentiable, and strictly decreasing on $[0, 1]$.

PROOF. Let

$$H_N(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{\sum_{m=-N}^{N} \operatorname{sgn}(m)(z-m)^{-2} + 2z^{-1}\right\}$$

so that

$$\lim_{N\to\infty} H_N(z) = H(z) \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{2} H_N'(z) = J(z)$$

uniformly on compact subsets of $\mathbf{C}$. From the identities

$$K(z) = \int_{-1}^{1} (1 - |t|)e(tz)\,dt \tag{2.29}$$

and

$$zK(z) = \frac{1}{2\pi i}\int_{-1}^{1} \operatorname{sgn}(t)e(tz)\,dt, \tag{2.30}$$

we have

$$\begin{aligned} H_N(z) &= \sum_{m=-N}^{N} \operatorname{sgn}(m)K(z-m) + 2zK(z) \\ &= \int_{-1}^{1} (1-|t|)\left\{\sum_{m=-N}^{N} \operatorname{sgn}(m)e(-mt)\right\}e(tz)\,dt \\ &\quad + \frac{1}{\pi i}\int_{-1}^{1} \operatorname{sgn}(t)e(tz)\,dt. \end{aligned} \tag{2.31}$$

As in our proof of Lemma 2, we apply $\frac{1}{2}(d/dz)$ to both sides of (2.31), use the identity (2.12) and the Riemann-Lebesgue lemma, and conclude that

$$J(z) = \int_{-1}^{1} \{\pi t(1-|t|)\cot \pi t + |t|\}e(tz)\,dt. \tag{2.32}$$

Let $\varphi(t) = \pi t(1-t)\cot \pi t + t$ for $-1 < t < 2$, with φ defined by continuity at 0 and 1. We then write (2.32) in the form

$$J(z) = 2\int_{0}^{1} \varphi(t)\cos(2\pi tz)\,dt \tag{2.33}$$

and integrate by parts three times. This provides the representation

$$J(z) = \frac{1}{(2\pi z)^3}\left\{2\int_0^1 \varphi'''(t)\sin(2\pi tz)\,dt - \frac{4\pi^2}{3}\sin 2\pi z\right\}$$

and also shows that (2.27) must hold. It is clear from (2.27) that $J(x)$ is integrable and (2.28) follows from (2.32). The remaining properties attributed to $\hat{J}(t)$ are all easily verified.

COROLLARY 7. *The Fourier transform of the function* $E(x) = H(x) - \operatorname{sgn}(x)$ *is given by*

$$\hat{E}(t) = \begin{cases} 0 & \textit{if } t = 0, \\ (\pi i t)^{-1}\{\hat{J}(t) - 1\} & \textit{if } t \neq 0. \end{cases} \tag{2.34}$$

We are now in a position to prove Beurling's result that the function $B(z)$ is extremal in inequality (1.4). In §3 we show that the representation (2.22) also holds when $F(z)$ has exponential type 2π and is integrable on $\mathbf{R}$. We use this fact in our proof of Beurling's theorem.

THEOREM 8. *Let $F(z)$ be an entire function of exponential type σ such that $F(x) \geqslant \operatorname{sgn}(x)$ for all real x. If $\sigma > 0$ then*

$$\int_{-\infty}^{\infty} F(x) - \operatorname{sgn}(x)\,dx \geqslant \frac{2\pi}{\sigma}; \tag{2.35}$$

if $\sigma = 0$ *then*

$$\int_{-\infty}^{\infty} F(x) - \operatorname{sgn}(x)\,dx = +\infty. \tag{2.36}$$

Moreover, there is equality in (2.35) *if and only if* $F(z) = B(\sigma(2\pi)^{-1}z)$.

PROOF. If $\sigma > 0$ then without loss of generality we may suppose that $\sigma = 2\pi$. We proceed exactly as in the proof of Theorem 4, but use H and J in place of G and I. We let $\psi(x) = F(x) - \operatorname{sgn}(x)$, $\varphi(x) = \frac{1}{2}F'(x)$, and show that $\varphi(x)$ is integrable. Again we obtain the identity

$$\hat{\psi}(t) = (\pi i t)^{-1}\{\hat{\varphi}(t) - 1\}$$

if $t \neq 0$. Since $\hat{\varphi}(t)$ is now supported on the interval $[-1, 1]$, we have

$$\hat{\psi}(t) = -(\pi i t)^{-1} \tag{2.37}$$

for $|t| \geqslant 1$.

At this point we use the Poisson summation formula (Zygmund [**Zyg,** Vol. I, p. 68]). Specifically, $\psi(x)$ is a normalized function of bounded variation; therefore

$$\sum_{l=-\infty}^{\infty} \psi(x + l) = \sum_{m=-\infty}^{\infty} \hat{\psi}(m)e(mx) \tag{2.38}$$

at each point x (the symmetric partial sums of both series converge to the same value). From (2.37) we obtain

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \hat{\psi}(m)e(mx) &= \hat{\psi}(0) - \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (\pi i m)^{-1}e(mx) \\ &= \begin{cases} \hat{\psi}(0) + 2(x - [x] - 1/2) & \text{if } x \notin \mathbf{Z}, \\ \hat{\psi}(0) & \text{if } x \in \mathbf{Z}. \end{cases} \end{aligned} \tag{2.39}$$

Now we observe that $\psi(x) \geqslant 0$, and, hence, the function (2.38) is nonnegative. This obviously implies that $\hat{\psi}(0) \geqslant 1$, which is exactly inequality (2.35).

If there is equality in (2.35)—that is, if $\hat{\psi}(0) = 1$—then

$$\lim_{x\to 0+} \sum_{l=-\infty}^{\infty} \psi(x + l) = 0.$$

Hence, $F(l+) = \operatorname{sgn}(l+)$, and therefore $F(l) = B(l)$, at each integer l. Since $F(x)$ majorizes $\operatorname{sgn}(x)$, we also conclude that $F'(l) = 0 = B'(l)$ at each integer $l \neq 0$. When we expand the entire function $F(z) - B(z)$, which is integrable on $\mathbf{R}$, by using (2.22), we find that

$$F(z) - B(z) = (F'(0) - 2)zK(z).$$

Since $xK(x)$ is *not* integrable, we must have $F'(0) = 2$; hence, $F(z) = B(z)$.

Finally, if $\sigma = 0$ we deduce that (2.36) holds exactly as in our proof of Theorem 4.

3. Interpolation formulas. The representation (2.22) is useful for constructing majorants because it allows us to control both $F(n)$ and $F'(n)$. We now give a general account of this interpolation formula. Throughout this section $F(z)$

will denote an entire function of exponential type σ with $\sigma \leqslant 2\pi$. For $0 < p \leqslant \infty$ let E^p be the set of those functions $F(z)$ which satisfy

$$\int_{-\infty}^{\infty} |F(x)|^p \, dx < \infty \quad \text{for } 0 < p < \infty,$$

and

$$\sup_{-\infty < x < \infty} |F(x)| < \infty \quad \text{for } p = \infty.$$

If $F(z)$ is in E^p, $0 < p < \infty$, then, by a result of Plancherel and Polya **[P-P]**,

$$\sum_{m=-\infty}^{\infty} |F(m)|^p \ll_p \int_{-\infty}^{\infty} |F(x)|^p \, dx \tag{3.1}$$

and

$$\int_{-\infty}^{\infty} |F'(x)|^p \, dx \ll_p \int_{-\infty}^{\infty} |F(x)|^p \, dx. \tag{3.2}$$

For $p = \infty$ the analogue of (3.2) is given by the classical inequality of Bernstein **[Brn]**: if $F(z)$ is in E^∞ then

$$\sup_{-\infty < x < \infty} |F'(x)| \leqslant 2\pi \Big\{ \sup_{-\infty < x < \infty} |F(x)| \Big\}. \tag{3.3}$$

By the Paley-Weiner theorem, $F(z)$ is in E^2 if and only if

$$F(z) = \int_{-1}^{1} \hat{F}(t) e(tz) \, dt \tag{3.4}$$

with $\hat{F}(t)$ in $L^2([-1, 1])$. Of course, $\hat{F}(t)$ is the Fourier transform of F, defined for almost all t by

$$\hat{F}(t) = \lim_{T \to \infty} \int_{-T}^{T} F(x) e(-tx) \, dx$$

and $\hat{F}(t) = 0$ for almost all t, with $|t| \geqslant 1$.

THEOREM 9. *Let $F(z)$ be an entire function in E^p for some finite p. Then*

$$F(z) = \left(\frac{\sin \pi z}{\pi} \right)^2 \left\{ \sum_{m=-\infty}^{\infty} F(m)(z - m)^{-2} + \sum_{n=-\infty}^{\infty} F'(n)(z - n)^{-1} \right\}, \tag{3.5}$$

where the expression on the right of (3.5) *converges uniformly on compact subsets of* **C**.

If $p = 2$ the Fourier transform $\hat{F}(t)$ occurring in (3.4) *has the form*

$$\hat{F}(t) = (1 - |t|) u_F(t) + (2\pi i)^{-1} \operatorname{sgn}(t) v_F(t) \tag{3.6}$$

for almost all t in $[-1, 1]$, *where u_F and v_F are periodic functions in* $L^2([0, 1])$ *with period* 1 *and Fourier series expansions*

$$u_F(t) = \sum_{m=-\infty}^{\infty} F(m) e(-mt) \tag{3.7}$$

and

$$v_F(t) = \sum_{n=-\infty}^{\infty} F'(n) e(-nt). \tag{3.8}$$

If $p = 1$ then (3.7) *and* (3.8) *are absolutely convergent, u_F and v_F are continuous, and* (3.6) *holds for all t in* $[-1, 1]$. *In particular,*

$$\sum_{m=-\infty}^{\infty} F(m) = u_F(0) = \hat{F}(0) \tag{3.9}$$

and

$$\sum_{n=-\infty}^{\infty} F'(n) = v_F(0) = 0. \tag{3.10}$$

PROOF. To begin with, we suppose that $p = 2$, so $F(z)$ is given by (3.4). For $0 \leqslant t < 1$ we define

$$u_F(t) = \hat{F}(t) + \hat{F}(t-1) \tag{3.11}$$

and

$$v_F(t) = 2\pi i\{t\hat{F}(t) + (t-1)\hat{F}(t-1)\}. \tag{3.12}$$

We then extend the domain of u_F and v_F to $\mathbf{R}$ by requiring that both functions have period 1. Since $\hat{F}$ is in $L^2([-1, 1])$, it is clear that u_F and v_F are in $L^2([0, 1])$. The identity (3.6) follows easily from (3.11), (3.12), and the periodicity of u_F and v_F. To obtain the expansions (3.7) and (3.8), we note that

$$F(n) = \int_0^1 \{\hat{F}(t) + \hat{F}(t-1)\} e(tn)\, dt = \int_0^1 u_F(t) e(tn)\, dt$$

and

$$\begin{aligned} F'(n) &= \int_{-1}^1 2\pi i t\hat{F}(t) e(tn)\, dt \\ &= \int_0^1 2\pi i\{t\hat{F}(t) + (t-1)\hat{F}(t-1)\} e(tn)\, dt \\ &= \int_0^1 v_F(t) e(tn)\, dt \end{aligned}$$

for each integer n. Thus, $F(n)$ and $F'(n)$ are the Fourier coefficients of u_F and v_F, respectively.

Next we apply the Fourier transform identities (2.29) and (2.30). It follows that for each positive integer N,

$$\begin{aligned} &\left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-N}^{N} F(m)(z-m)^{-2} + \sum_{n=-N}^{N} F'(n)(z-n)^{-1} \right\} \\ &\quad = \int_{-1}^1 \left\{(1-|t|) u_F(t, N) + (2\pi i)^{-1} \operatorname{sgn}(t) v_F(t, N) \right\} e(tz)\, dt, \end{aligned} \tag{3.13}$$

where

$$u_F(t, N) = \sum_{m=-N}^{N} F(m) e(-mt)$$

and

$$v_F(t, N) = \sum_{n=-N}^{N} F'(n) e(-nt).$$

Since the sequences $F(m)$ and $F'(n)$ are square summable, the left side of (3.13) converges uniformly on compact subsets of $\mathbf{C}$ as $N \to \infty$. On the right side of (3.13) we have $u_F(t, N) \to u_F(t)$ and $v_F(t, N) \to v_F(t)$ in L^2-norm. This is all we need to establish the representation (3.5).

If $p = 1$ then (3.1) and (3.2) imply that $u_F(t)$ and $v_F(t)$ have absolutely convergent Fourier series. Thus, we may take u_F and v_F to be continuous periodic functions. Since $\hat{F}(t)$ is now continuous and supported on $[-1, 1]$, the identity (3.6) must hold for all t in $[-1, 1]$. If we let $t = 0$, then (3.9) and (3.10) follow immediately.

Finally, we must show that (3.5) holds if $F(z)$ is an entire function in E^p with $2 < p < \infty$. We accomplish this by considering the entire function

$$(3.14) \qquad R(z) = \begin{cases} z^{-1}(F(z) - F(0)) & \text{if } z \neq 0, \\ F'(0) & \text{if } z = 0 \end{cases}$$

and its derivative

$$(3.15) \qquad R'(z) = \begin{cases} z^{-2}(F(0) + zF'(z) - F(z)) & \text{if } z \neq 0, \\ \frac{1}{2}F''(0) & \text{if } z = 0. \end{cases}$$

Since $R(z)$ is in E^2, by the first part of our proof we have

$$(3.16) \qquad R(z) = \lim_{N \to \infty} \left(\frac{\sin \pi z}{\pi}\right)^2 \cdot \left\{ \sum_{m=-N}^{N} R(m)(z - m)^{-2} + \sum_{n=-N}^{N} R'(n)(z - n)^{-1} \right\}$$

uniformly on compact subsets of $\mathbf{C}$. Next we multiply both sides of (3.16) by z and use (3.14) and (3.15). After a brief computation we find that

$$(3.17) \qquad F(z) - F(0) = \lim_{N \to \infty} \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-N}^{N} F(m)(z - m)^{-2} + \sum_{n=-N}^{N} F'(n)(z - n)^{-1} + \sum_{k=-N}^{N} R'(k) - F(0) \sum_{l=-N}^{N} (z - l)^{-2} \right\}.$$

As the identity

$$\sum_{l=-\infty}^{\infty} (z - l)^{-2} = \left(\frac{\pi}{\sin \pi z}\right)^2$$

is well known; all that remains is to show that

$$\lim_{N \to \infty} \sum_{k=-N}^{N} R'(k) = 0.$$

We have

$$(3.18) \qquad \int_{1 \leqslant |x|} |R'(x)|\, dx \leqslant \int_{1 \leqslant |x|} |x|^{-2} |F(0) - F(x)|\, dx + \int_{1 \leqslant |x|} |x|^{-1} |F'(x)|\, dx.$$

The first integral on the right of (3.18) is obviously finite. The second integral is also finite because $F'(x)$ is in $L^p(\mathbf{R})$ by (3.2). Thus, $R'(z)$ is in E^1. Since we have already established (3.10) for functions in E^1, we obtain

$$\sum_{k=-\infty}^{\infty} R'(k) = \int_{-\infty}^{\infty} R'(x)\,dx = 0.$$

This completes our proof.

THEOREM 10. *Let $F(z)$ be an entire function of exponential type σ with $\sigma \leqslant 2\pi$, let $R(z)$ be defined by* (3.14), *and suppose that $R(z)$ is in E^p for some finite p. Then*

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-\infty}^{\infty} F(m)(z-m)^{-2} + F'(0)z^{-1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} F'(n)\left\{(z-n)^{-1} + n^{-1}\right\} + A_F \right\}, \tag{3.19}$$

where the expression on the right of (3.19) *converges uniformly on compact subsets of* $\mathbf{C}$, *and A_F is a constant given by*

$$A_F = \frac{1}{2}F''(0) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (F(0) - F(n))n^{-2}. \tag{3.20}$$

PROOF. Since $R(z)$ is in E^p, we may apply Theorem 9. As in the proof of that result, we find that (3.16) and (3.17) holds. Now, however, we reorganize (3.17) and use (3.15) to obtain

$$\begin{aligned} F(z) = \lim_{N\to\infty} \left(\frac{\sin \pi z}{\pi}\right)^2 & \cdot \left\{ \sum_{m=-N}^{N} F(m)(z-m)^{-2} + F'(0)z^{-1} + \sum_{\substack{n=-N \\ n \neq 0}}^{N} F'(n)\left\{(z-n)^{-1} + n^{-1}\right\} \right. \\ & \left. + \frac{1}{2}F''(0) + \sum_{\substack{k=-N \\ k \neq 0}}^{N} (F(0) - F(k))k^{-2} \right\}. \end{aligned} \tag{3.21}$$

If $0 < p < q < \infty$, then $E^p \subseteq E^q$. Thus, we may assume without loss of generality that $1 < p < \infty$. It follows that

$$\begin{aligned} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left|F(m)m^{-1}\right|^p &= \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left|R(m) + F(0)m^{-1}\right|^p \\ &\leqslant 2^p \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left\{\left|R(m)\right|^p + \left|F(0)m^{-1}\right|^p\right\} < \infty, \end{aligned} \tag{3.22}$$

and

$$(3.23)\qquad \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} |F'(n)n^{-1}|^p = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} |R'(n)+R(n)n^{-1}|^p$$
$$\leqslant 2^p \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \left\{|R'(n)|^p + |R(n)n^{-1}|^p\right\} < \infty.$$

For the series defining A_F we also have

$$(3.24)\qquad \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} |(F(0)-F(k))k^{-2}| = \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} |R(k)|\,|k|^{-1} < \infty.$$

Estimates (3.22)–(3.24), together with (3.21), show that the right side of (3.19) converges uniformly on compact subsets, with A_F given by the absolutely convergent series (3.20).

We note that if $F(z)$ is odd and bounded on $\mathbf{R}$, then $A_F = 0$ and (3.19) reduces to (2.22).

4. Majorizing functions of bounded variation. Let $f\colon \mathbf{R} \to \mathbf{C}$ be a normalized function of bounded variation. The results of §2 can be applied in a simple way to prove approximation and majorization theorems for such a function f. We use the following notation: If $\delta > 0$ we write $F_\delta(x) = \delta F(\delta x)$. If $F(x)$ is integrable we have $\hat{F}_\delta(t) = \hat{F}(\delta^{-1}t)$. In this case we also define the convolutions

$$f * F(x) = \int_{-\infty}^{\infty} f(\xi)F(x-\xi)\,d\xi$$

and

$$(df) * F(x) = \int_{-\infty}^{\infty} F(x-\xi)\,df(\xi).$$

For entire functions $F(z)$ of exponential type σ with $F(x)$ integrable, it is easy to verify that $f * F(z)$ and $(df) * F(z)$ have exponential type at most σ. We denote the total variation of f on $(-\infty, x]$ by $V_f(x)$ and let $V_f = \lim_{x\to+\infty} V_f(x)$.

THEOREM 11. *The entire function $f * I_\delta(z)$ has exponential type at most $\pi\delta$ and satisfies*

$$(4.1)\qquad \int_{-\infty}^{\infty} |f(x) - f * I_\delta(x)|\,dx \leqslant (2\delta)^{-1}V_f.$$

The entire functions $f * J_\delta(z)$ and $(dV_f) * K_\delta(z)$ have exponential type at most $2\pi\delta$ and satisfy

$$(4.2)\qquad |f(x) - f * J_\delta(x)| \leqslant (2\delta)^{-1}(dV_f) * K_\delta(x)$$

for all real x.

PROOF. Let x be a point at which f is continuous. Then

$$\begin{aligned} f(x) &= \frac{1}{2}\int_{-\infty}^{\infty} f(x-\xi)\, d\{\operatorname{sgn}(\delta\xi)\} = \int_{-\infty}^{\infty} f(x-\xi) I_\delta(\xi)\, d\xi \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} f(x-\xi)\, d\{\operatorname{sgn}(\delta\xi) - G(\delta\xi)\} \\ &= f * I_\delta(x) + \frac{1}{2}\int_{-\infty}^{\infty} \{\operatorname{sgn}(\delta(x-\xi)) - G(\delta(x-\xi))\}\, df(\xi) \end{aligned}$$

after integrating by parts. Since f is continuous at almost all x, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} |f(x) - f * I_\delta(x)|\, dx \\ &\quad \leqslant \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\operatorname{sgn}(\delta(x-\xi)) - G(\delta(x-\xi))|\, dV_f(\xi)\, dx \\ &\quad = (2\delta)^{-1} V_f, \end{aligned}$$

by using the case of equality in (2.16).

In like manner we obtain

$$f(x) - f * J_\delta(x) = \frac{1}{2}\int_{-\infty}^{\infty} \{\operatorname{sgn}(\delta(x-\xi)) - H(\delta(x-\xi))\}\, df(\xi) \tag{4.3}$$

at each point x where f is continuous. By combining (2.24) and (4.3) we find that

$$\begin{aligned} |f(x) - f * J_\delta(x)| &\leqslant \frac{1}{2\delta}\int_{-\infty}^{\infty} K_\delta(x-\xi)\, dV_f(\xi) \\ &= \frac{1}{2\delta}(dV_f) * K_\delta(x). \end{aligned} \tag{4.4}$$

Since f satisfies the normalization (1.16), it is trivial to extend (4.4) to points of discontinuity by considering left- and right-hand limits. This completes the proof.

The Fourier transforms of the integrable functions

$$\Psi_\delta(f, x) = f(x) - f * I_\delta(x)$$

and

$$\Phi_\delta(f, x) = f(x) - f * J_\delta(x)$$

can be computed exactly as in our proof of Corollary 3. We find that

$$\hat{\Psi}_\delta(f, t) = \widehat{df}(t)\left(\frac{1 - \hat{I}(\delta^{-1}t)}{2\pi i t}\right) \tag{4.5}$$

and

$$\hat{\Phi}_\delta(f, t) = \widehat{df}(t)\left(\frac{1 - \hat{J}(\delta^{-1}t)}{2\pi i t}\right) \tag{4.6}$$

for $t \neq 0$, and $\hat{\Psi}_\delta(f, 0) = \hat{\Phi}_\delta(f, 0) = 0$. Here we have written

$$\widehat{df}(t) = \int_{-\infty}^{\infty} e(-tx)\, df(x) \tag{4.7}$$

for the Fourier-Stieltjes transform of f.

For real-valued functions f, (4.2) provides the following majorization.

COROLLARY 12. *If f is real valued then the entire function*

$$M(f,\delta,z)=f*J_\delta(z)+(2\delta)^{-1}(dV_f)*K_\delta(z) \tag{4.8}$$

has exponential type at most $2\pi\delta$, satisfies $f(x)\leqslant M(f,\delta,x)$ for all real x, and

$$\int_{-\infty}^{\infty} M(f,\delta,x)-f(x)\,dx=(2\delta)^{-1}V_f. \tag{4.9}$$

Since the total variation of $\operatorname{sgn}(x)$ is 2, Theorem 4 shows that there is equality in (4.1) when $f(x)=\operatorname{sgn}(x)$.

If $f(x)$ is real valued and integrable, then $M(f,\delta,x)$ is integrable. A straightforward computation shows that

$$\hat{M}(f,\delta,t)=\hat{f}(t)\hat{J}_\delta(t)+(2\delta)^{-1}\widehat{dV_f}(t)\hat{K}_\delta(t)$$

for all real t. Thus $\hat{M}(f,\delta,t)$ is supported on $[-\delta,\delta]$. In particular, if $f(x)=\chi_E(x)$ for some interval $E=[\alpha,\beta]$, then the right side of (4.8) is

$$\begin{aligned}\frac{1}{2}\int_\alpha^\beta \delta H'(\delta(x-\xi))\,d\xi &+\frac{1}{2\delta}\{K_\delta(x-\alpha)+K_\delta(x-\beta)\}\\ &=\frac{1}{2}\{H(\delta(x-\alpha))-H(\delta(x-\beta))\}\\ &\quad+\frac{1}{2}\{K(\delta(x-\alpha))+K(\delta(x-\beta))\}\\ &=\frac{1}{2}\{B(\delta(\beta-x))+B(\delta(x-\alpha))\}.\end{aligned}$$

Thus Corollary 12 generalizes Selberg's construction (1.5).

While the function $M(f,\delta,z)$ majorizes $f(x)$ on $\mathbf{R}$, we cannot, in general, expect $M(f,\delta,z)$ to be an extremal function which minimizes the integral (4.9). For certain special functions f, such an extreme majorant can be obtained by simply constructing, by means of Theorems 9 or 10, an entire function of exponential type which interpolates $f(x)$ and $f'(x)$ at the integers. This approach has been carried out by Graham and Vaaler [$\mathbf{GV_2}$]. An alternative method is illustrated by the entire functions $\tau_k(z)=z^{2k-1}H(z)$, where $k=0,1,2,\ldots$. It is clear that each $\tau_k(z)$ has exponential type 2π and, by (2.23), satisfies $\tau_k(x)\leqslant |x|^{2k-1}$ for all real x. For $k=0$ this can be improved to $\tau_0(x)\leqslant \min\{2,|x|^{-1}\}$. Among all entire functions of type 2π which minorize $\min\{2,|x|^{-1}\}$, the function $\tau_0(z)$ can be shown to be extremal in the sense that $\int_{-\infty}^{\infty}\min\{2,|x|^{-1}\}-\tau_0(x)\,dx$ is minimized. We note that $\tau_0(z)$ interpolates the values of $\min\{2,|x|^{-1}\}$ and its derivative at the integers. For $k=1,2,\ldots$ the nonnegative functions $|x|^{2k-1}-\tau_k(x)$ are no longer integrable.

5. The Berry-Esseen inequality. Let $f(x)$ and $g(x)$ be normalized probability distribution functions. An important problem in probability theory is to estimate the difference $f(x)-g(x)$ by an expression depending on $\widehat{df}(t)-\widehat{dg}(t)$ for values of t restricted to an interval $|t|\leqslant\delta$. For example, if $g(x)$ has a density function bounded from above by the positive constant M, then

$$\sup_x |f(x) - g(x)| \leqslant c_1\delta^{-1}M + c_2\int_{-\delta}^{\delta} |t|^{-1}\left|\widehat{df}(t) - \widehat{dg}(t)\right| dt \tag{5.1}$$

for certain positive constants c_1 and c_2 and all $\delta > 0$. With our definition of the Fourier-Stieltjes transform, the usual proofs of (5.1) give $c_1 = 12\pi^{-2}$ and $c_2 = \pi^{-1}$ (Feller [**Fel**], Loève [**Loe**]). Earlier versions of this inequality were used by Berry [**Brr**] and Esseen [**Ess**] in order to estimate the rate of convergence in the central limit theorem. Since their work has appeared, there have been many refinements. We note in particular the papers of Zolotarev [**Zo$_1$-Zo$_3$**], van Beek [**vBe**], and H. Prawitz [**Pr$_1$-Pr$_5$**].

Inequalities sharper than (5.1) can be obtained directly from Corollary 12. Here we give two results of this type which are essentially the same as those obtained by Prawitz [**Pr$_1$**]. For the purpose of this application, Prawitz obtained inequality (2.24) in a manner quite different from ours.

THEOREM 13. *Let $f(x)$ and $g(x)$ be probability distribution functions with Fourier-Stieltjes transforms defined by* (4.7). *Suppose $t^{-1}\{\widehat{df}(t) - \widehat{dg}(t)\}$ is integrable on a neighborhood of zero and g has a density function $g'(x)$ bounded from above by M. Then*

$$\left|f(x) - g(x) - \int_{-\delta}^{\delta} \hat{J}(\delta^{-1}t)(2\pi i t)^{-1}\left\{\widehat{df}(t) - \widehat{dg}(t)\right\} e(xt)\, dt\right| \leqslant \frac{1}{2\delta}\left\{M + \int_{-\delta}^{\delta} \hat{K}(\delta^{-1}t)\left\{\widehat{df}(t) - \widehat{dg}(t)\right\} e(xt)\, dt\right\} \tag{5.2}$$

for all real x and all $\delta > 0$.

PROOF. Since $f(x)$ is increasing, (4.7) takes the form

$$f(x) \leqslant f * J_\delta(x) + (2\delta)^{-1}(df) * K_\delta(x).$$

It follows that

$$\begin{aligned} f(x) - g(x) \leqslant{}& (f - g) * J_\delta(x) + (2\delta)^{-1}(df - dg) * K_\delta(x) \\ &+ g * J_\delta(x) + (2\delta)^{-1}(dg) * K_\delta(x) - g(x). \end{aligned} \tag{5.3}$$

In the first term on the right of (5.3) we have

$$\begin{aligned} (f - g) * J_\delta(x) &= \int_{-\infty}^{\infty} \{f(\xi) - g(\xi)\} J_\delta(x - \xi)\, d\xi \\ &= \frac{1}{2}\int_{-\infty}^{\infty} H(\delta(x - \xi))\, d\{f(\xi) - g(\xi)\} \\ &= \int_{-\infty}^{\infty} \left(\int_u^x J_\delta(\omega - \xi)\, d\omega\right) d\{f(\xi) - g(\xi)\} \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} H(\delta(u - \xi))\, d\{f(\xi) - g(\xi)\}. \end{aligned} \tag{5.4}$$

Here u is an arbitrary real number less than x. Applying the Fourier inversion formula to J, we obtain

$$(5.5)\qquad \int_u^x J_\delta(\omega-\xi)\,d\omega = \int_u^x\int_{-\delta}^{\delta}\hat{J}_\delta(t)e((\omega-\xi)t)\,dt\,d\omega$$
$$= \int_{-\delta}^{\delta}\hat{J}_\delta(t)\left\{\int_u^x e(\omega t)\,d\omega\right\}e(-\xi t)\,dt.$$

Using Fubini's theorem we find that

$$(5.6)\qquad \int_{-\infty}^{\infty}\left(\int_u^x J_\delta(\omega-\xi)\,d\omega\right)d\{f(\xi)-g(\xi)\}$$
$$= \int_{-\delta}^{\delta}\hat{J}_\delta(t)\left\{\int_u^x e(\omega t)\,d\omega\right\}\left\{\int_{-\infty}^{\infty}e(-\xi t)\,d\{f(\xi)-g(\xi)\}\right\}dt$$
$$= \int_{-\delta}^{\delta}\hat{J}_\delta(t)\{e(xt)-e(ut)\}(2\pi it)^{-1}\{\widehat{df}(t)-\widehat{dg}(t)\}\,dt.$$

Since the function $(2\pi it)^{-1}\{\widehat{df}(t)-\widehat{dg}(t)\}$ is integrable on $[-\delta,\delta]$, we may combine (5.4) and (5.6) in the form

$$(5.7)\qquad (f-g)*J_\delta(x) = \int_{-\delta}^{\delta}\hat{J}_\delta(t)(2\pi it)^{-1}\{\widehat{df}(t)-\widehat{dg}(t)\}e(xt)\,dt$$
$$-\int_{-\delta}^{\delta}\hat{J}_\delta(t)(2\pi it)^{-1}\{\widehat{df}(t)-\widehat{dg}(t)\}e(ut)\,dt$$
$$+\frac{1}{2}\int_{-\infty}^{\infty}H(\delta(u-\xi))\,d\{f(\xi)-g(\xi)\}.$$

We now let $u\to-\infty$. The second integral on the right of (5.7) tends to zero by the Riemann-Lebesgue lemma. In the third integral on the right of (5.7) we use (2.24) and the dominated convergence theorem. We obtain

$$\lim_{u\to-\infty}\frac{1}{2}\int_{-\infty}^{\infty}H(\delta(u-\xi))\,d\{f(\xi)-g(\xi)\} = -\frac{1}{2}\int_{-\infty}^{\infty}d\{f(\xi)-g(\xi)\} = 0,$$

so

$$(5.8)\qquad (f-g)*J_\delta(x) = \int_{-\delta}^{\delta}\hat{J}_\delta(t)(2\pi it)^{-1}\{\widehat{df}(t)-\widehat{dg}(t)\}e(xt)\,dt.$$

Next we write the second term on the right of (5.3) as

$$(5.9)\qquad \frac{1}{2\delta}\int_{-\infty}^{\infty}K_\delta(x-\xi)\,d\{f(\xi)-g(\xi)\}$$
$$= \frac{1}{2\delta}\int_{-\delta}^{\delta}\hat{K}_\delta(t)\left\{\int_{-\infty}^{\infty}e((x-\xi)t)\,d\{f(\xi)-g(\xi)\}\right\}dt$$
$$= \frac{1}{2\delta}\int_{-\delta}^{\delta}\hat{K}_\delta(t)\{\widehat{df}(t)-\widehat{dg}(t)\}e(xt)\,dt.$$

Finally, the last three terms on the right of (5.3) can be estimated using (1.3), (4.3), and our bound on the density function $g'(x)$. We find that

$$(5.10)\qquad \begin{aligned} &g * J_\delta(x) + (2\delta)^{-1}(dg) * K_\delta(x) - g(x) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} B(\delta(x-\xi)) - \operatorname{sgn}(\delta(x-\xi))\, dg(\xi) \\ &\leqslant \frac{1}{2}M\int_{-\infty}^{\infty} B(\delta(x-\xi)) - \operatorname{sgn}(\delta(x-\xi))\, d\xi \\ &= (2\delta)^{-1}M. \end{aligned}$$

When we combine (5.8)–(5.10) we obtain an upper bound for $f(x) - g(x)$. Of course, (4.2) also provides a minorizing inequality

$$f(x) \geqslant f * J_\delta(x) - (2\delta)^{-1}(df) * K_\delta(x).$$

This leads to a corresponding lower bound for $f(x) - g(x)$. To complete the proof, the upper and lower bounds can be written together as (5.2).

By making a slight variation in our proof of Theorem 13 we get a similar inequality, but without the requirement that $g(x)$ have a bounded density.

THEOREM 14. *Let $f(x)$ and $g(x)$ be probability distribution functions such that $t^{-1}\{\widehat{df}(t) - \widehat{dg}(t)\}$ is integrable on a neighborhood of zero. Then*

$$(5.11)\qquad \begin{aligned} &\left| f(x) - g(x) - \int_{-\delta}^{\delta} \hat{J}(\delta^{-1}t)(2\pi i t)^{-1}\{\widehat{df}(t) - \widehat{dg}(t)\}\, e(xt)\, dt \right| \\ &\leqslant \frac{1}{2\delta}\int_{-\delta}^{\delta} \hat{K}(\delta^{-1}t)\{\widehat{df}(t) + \widehat{dg}(t)\}\, e(xt)\, dt \end{aligned}$$

for all real x and all $\delta > 0$.

PROOF. We apply (4.2), instead of (5.3), directly to the function $f(x) - g(x)$. Since f and g are increasing and $K_\delta(x)$ is nonnegative, we have

$$\begin{aligned} |f(x) - g(x) - (f-g) * J_\delta(x)| &\leqslant (2\delta)^{-1}(dV_{f-g}) * K_\delta(x) \\ &\leqslant (2\delta)^{-1}(df + dg) * K_\delta(x). \end{aligned}$$

The result now follows by using (5.8) and (5.9) as in our proof of Theorem 13.

6. Further applications. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers and

$$f(x) = \sum_{n=1}^{N} a(n) e(\lambda_n x)$$

an almost periodic trigonometric polynomial.

THEOREM 15. *If $|\lambda_n| \geqslant \delta > 0$ for $n = 1, 2, \ldots, N$, then*

$$(6.1)\qquad \sup_x |f(x)| \leqslant (4\delta)^{-1} \sup_\xi |f'(\xi)|.$$

If, in addition, $f(x)$ is real valued, then

$$\sup_x |f(x)| \leqslant (2\delta)^{-1} \sup_\xi f'(\xi). \tag{6.2}$$

Moreover, the constants $(4\delta)^{-1}$ and $(2\delta)^{-1}$ are asymptotically best possible as $N \to \infty$.

PROOF. We have

$$\begin{aligned} f * I_{2\delta}(x) &= \int_{-\infty}^{\infty} f(x-\xi) I_{2\delta}(\xi)\, d\xi \\ &= \sum_{n=1}^{N} a(n) e(\lambda_n x) \int_{-\infty}^{\infty} I_{2\delta}(\xi) e(-\lambda_n \xi)\, d\xi \\ &= \sum_{n=1}^{N} a(n) \hat{I}_{2\delta}(\lambda_n) e(\lambda_n x) = 0, \end{aligned}$$

since $\hat{I}_{2\delta}(t) = 0$ for $|t| \geqslant \delta$. It follows that

$$\begin{aligned} f(x) &= \frac{1}{2}\int_{-\infty}^{\infty} f(x-\xi)\, d\{\operatorname{sgn}(2\delta\xi)\} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} f(x-\xi)\, d\{\operatorname{sgn}(2\delta\xi) - G(2\delta\xi)\} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \{\operatorname{sgn}(2\delta\xi) - G(2\delta\xi)\} f'(x-\xi)\, d\xi. \end{aligned}$$

Therefore, by Theorem 4,

$$\begin{aligned} |f(x)| &\leqslant \Big(\sup_\xi |f'(\xi)|\Big)\Big(\frac{1}{2}\int_{-\infty}^{\infty} |\operatorname{sgn}(2\delta u) - G(2\delta u)|\, du\Big) \\ &= (4\delta)^{-1} \sup_\xi |f'(\xi)|. \end{aligned}$$

This proves (6.1).

Since $\hat{J}_\delta(t) = \hat{K}_\delta(t) = 0$ for $|t| \geqslant \delta$, we also have

$$f * J_\delta(x) = \delta^{-1} f' * K_\delta(x) = 0$$

for all real x. This implies that

$$\begin{aligned} f(x) &= \frac{1}{2}\int_{-\infty}^{\infty} \{\operatorname{sgn}(\delta\xi) - H(\delta\xi)\} f'(x-\xi)\, d\xi \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \{\operatorname{sgn}(\delta\xi) - H(\delta\xi) \pm K(\delta\xi)\} f'(x-\xi)\, d\xi. \end{aligned} \tag{6.3}$$

Now we suppose that $f(x)$ is real valued and use the minus sign on the right of (6.3). In view of the bound (2.24) we find that

$$\begin{aligned} f(x) &\geqslant \Big(\sup_\xi f'(\xi)\Big)\Big(\frac{1}{2}\int_{-\infty}^{\infty} \{\operatorname{sgn}(\delta u) - H(\delta u) - K(\delta u)\}\, du\Big) \\ &= -(2\delta)^{-1} \sup_\xi f'(\xi). \end{aligned} \tag{6.4}$$

Similarly, using the plus sign on the right of (6.3), we obtain

$$f(x) \leqslant (2\delta)^{-1} \sup_{\xi} f'(\xi)$$

for all real x. This establishes (6.2).

In order to show that (6.1) and (6.2) are sharp, let

$$(6.5) \quad k_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e(nx) = (N+1)^{-1} \left(\frac{\sin \pi(N+1)x}{\sin \pi x}\right)^2$$

denote the periodic Fejer kernel. Then let

$$(6.6) \qquad \psi(x) = \begin{cases} x - [x] - 1/2 & \text{if } x \notin \mathbf{Z}, \\ 0 & \text{if } x \in \mathbf{Z}. \end{cases}$$

It suffices to assume that $\delta = 1$; then the function

$$\begin{aligned} f_N(x) = \psi * k_N(x) &= \int_0^1 \psi(\xi) k_N(x - \xi)\, d\xi \\ &= - \sum_{\substack{n=-N \\ n \neq 0}}^{N} (2\pi i n)^{-1} \left(1 - \frac{|n|}{N+1}\right) e(nx) \end{aligned}$$

is real valued and satisfies the hypotheses of the theorem. We have

$$f_N'(x) = - \sum_{\substack{n=-N \\ n \neq 0}}^{N} \left(1 - \frac{|n|}{N+1}\right) e(nx) = 1 - k_N(x),$$

and hence the right side of (6.2) is

$$\frac{1}{2} \sup_{\xi} f_N'(\xi) = \frac{1}{2}.$$

Since $f_N(x) \to \psi(x)$ pointwise, we must also have

$$\lim_{N \to \infty} \left(\sup_{x} |f_N(x)| \right) = \frac{1}{2}.$$

The proof that (6.1) is sharp is essentially the same, except $\|x\| - 1/4$ is used in place of $\psi(x)$.

Inequality (6.1) is due to H. Bohr (see [**Sha**, p. 142]), and (6.2) was discovered by Beurling. In fact, this inequality motivated Beurling's construction of the extremal function $B(z) = H(z) + K(z)$. Theorem 15 can be extended to absolutely continuous functions $f(x)$ such that f and f' are in $L^\infty(\mathbf{R})$ and the spectrum of f does not intersect the open interval $(-2\pi\delta, 2\pi\delta)$. This extension for (6.1) is discussed by Shapiro [**Sha**, Chapter 7].

Next we consider a general form of Hilbert's inequality first obtained by Montgomery and Vaughan [**M-V**] (see also [**Mon**] and [**GV$_2$**]). Here we give a new and particularly simple proof of this result.

THEOREM 16 (MONTGOMERY AND VAUGHAN). *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers satisfying $|\lambda_m - \lambda_n| \geqslant \delta > 0$ whenever $m \neq n$, and let $a(1), \ldots, a(N)$*

be arbitrary complex numbers. Then

$$(6.7)\qquad \left|\sum_{m=1}^{N}\sum_{\substack{n=1\\ m\neq n}}^{N}\frac{a(m)\overline{a(n)}}{\lambda_m-\lambda_n}\right| \leqslant \pi\delta^{-1}\sum_{n=1}^{N}|a(n)|^2.$$

PROOF. Let $\varphi(x) = H(x) + K(x) - \operatorname{sgn}(x)$ so that $\varphi(x)$ is nonnegative and integrable. Using (2.29) and (2.34) we see that

$$\hat{\varphi}(t) = \begin{cases} 1 & \text{if } t = 0,\\ -(\pi i t)^{-1} & \text{if } |t| \geqslant 1.\end{cases}$$

It follows that

$$\begin{aligned} 0 &\leqslant \int_{-\infty}^{\infty}\varphi_\delta(x)\left|\sum_{m=1}^{N}a(m)e(-\lambda_m x)\right|^2 dx\\ &= \sum_{m=1}^{N}\sum_{n=1}^{N}a(m)\overline{a(n)}\,\hat{\varphi}_\delta(\lambda_m-\lambda_n)\\ &= \sum_{n=1}^{N}|a(n)|^2\hat{\varphi}_\delta(0) - \sum_{m=1}^{N}\sum_{\substack{n=1\\ m\neq n}}^{N}a(m)\overline{a(n)}\left\{\pi i\delta^{-1}(\lambda_m-\lambda_n)\right\}^{-1}, \end{aligned}$$

and therefore

$$(6.8)\qquad \sum_{m=1}^{N}\sum_{\substack{n=1\\ m\neq n}}^{N}\frac{a(m)\overline{a(n)}}{i(\lambda_m-\lambda_n)} \leqslant \pi\delta^{-1}\sum_{n=1}^{N}|a(n)|^2.$$

If we begin with $H(x) - K(x) - \operatorname{sgn}(x)$, we obtain (6.8) with the inequality reversed and a factor of -1 on the right side. These bounds prove the theorem.

7. Majorizing periodic functions. We define periodic trigonometric polynomials $i_N(x), j_N(x)$, and $k_N(x)$ as follows:

$$(7.1)\qquad i_N(x) = \sum_{m=-\infty}^{\infty} I_{2N+2}(x+m) = \sum_{n=-N}^{N}\hat{I}_{2N+2}(n)e(nx),$$

$$(7.2)\qquad j_N(x) = \sum_{m=-\infty}^{\infty} J_{N+1}(x+m) = \sum_{n=-N}^{N}\hat{J}_{N+1}(n)e(nx),$$

$$(7.3)\qquad k_N(x) = \sum_{m=-\infty}^{\infty} K_{N+1}(x+m) = \sum_{n=-N}^{N}\hat{K}_{N+1}(n)e(nx).$$

Of course, k_N is the periodic Fejer kernel also given by (6.5). The identities indicated on the right of (7.1)–(7.3) follow immediately from the Poisson summation formula. For the purpose of approximating or majorizing periodic functions by trigonometric polynomials, we use i_N, j_N, and k_N in a role analogous to that played by the entire functions I, J, and K in Theorem 11. In the periodic case $\psi(x)$, defined by (6.6), can be used in place of $\operatorname{sgn}(x)$. Thus, we assume throughout this section that ψ is so defined. Here we use $f * g(x)$ to denote the convolution

$$f * g(x) = \int_{-1/2}^{1/2} f(x-\xi)g(\xi)\,d\xi$$

of two periodic functions with period 1. We also write

$$\hat{f}(n) = \int_{-1/2}^{1/2} f(x)e(-nx)\,dx$$

for the nth Fourier coefficient of the periodic function $f(x)$.

THEOREM 17. *The trigonometric polynomial*

$$\psi * i_N(x) = \sum_{\substack{n=-N \\ n\neq 0}}^{N} (-2\pi i n)^{-1}\hat{I}_{2N+2}(n)e(nx) \tag{7.4}$$

satisfies

$$\operatorname{sgn}(\psi * i_N(x)) = \operatorname{sgn}(\psi(x)), \tag{7.5}$$

$$\operatorname{sgn}(\psi * i_N(x) - \psi(x)) = \operatorname{sgn}(\sin 2\pi(N+1)x), \tag{7.6}$$

and

$$\int_{-1/2}^{1/2} |\psi * i_N(x) - \psi(x)|\,dx = (4N+4)^{-1}. \tag{7.7}$$

If $p_N(x)$ is any trigonometric polynomial of degree at most N, then

$$\int_{-1/2}^{1/2} |p_N(x) - \psi(x)|\,dx \geqslant (4N+4)^{-1}. \tag{7.8}$$

Moreover, there is equality in (7.8) *if and only if $p_N(x) = \psi * i_N(x)$.*

PROOF. We have

$$\begin{aligned}\psi * i_N(x) &= -\sum_{n=1}^{N} \hat{I}_{2N+2}(n)\left(\frac{\sin 2\pi n x}{\pi n}\right) \\ &= -\sum_{n=1}^{N} \{\hat{I}_{2N+2}(n) - \hat{I}_{2N+2}(n+1)\}S(x,n),\end{aligned} \tag{7.9}$$

where

$$S(x,n) = \sum_{l=1}^{n} \frac{\sin 2\pi l x}{\pi l}.$$

By a classical result of Fejer [**Fej**, Satz XXVI] the partial sums $S(x,n)$ are positive for $0 < x < 1/2$. Since $\hat{I}_{2N+2}(n)$ is strictly decreasing for $n = 1, 2, \ldots, N+1$ (by (2.10)), the right side of (7.9) is negative for $0 < x < 1/2$. As $\psi * i_N(x)$ is odd, continuous, and periodic, this proves (7.5) for all real x.

Let $D(x)$ be defined as in Corollary 3 and write $\delta = 2N+2$. Then

$$\begin{aligned}-\frac{1}{2\delta}\sum_{m=-\infty}^{\infty} D_\delta(x+m) &= -\frac{1}{2\delta}\sum_{n=-\infty}^{\infty} \hat{D}_\delta(n)e(nx) \\ &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} -\frac{1}{2\pi i n}\{\hat{I}_{2N+2}(n) - 1\}e(nx) = \psi * i_N(x) - \psi(x)\end{aligned} \tag{7.10}$$

by (2.15) and the Poisson summation formula. Using (2.4) we have

$$\text{(7.11)}\qquad \begin{aligned}\operatorname{sgn}\left\{-(2\delta)^{-1}D_\delta(x+m)\right\} &= \operatorname{sgn}\{\sin 2\pi(N+1)(x+m)\}\\ &= \operatorname{sgn}\{\sin 2\pi(N+1)x\}\end{aligned}$$

for all real x and every integer m. This clearly implies (7.6).

To establish (7.7) we combine (2.16), (7.10), and (7.11). We find that

$$\begin{aligned}\int_{-1/2}^{1/2} |\psi * i_N(x) - \psi(x)|\,dx &= \frac{1}{2\delta}\int_{-1/2}^{1/2}\left|\sum_{m=-\infty}^{\infty} D_\delta(x+m)\right| dx\\ &= \frac{1}{2\delta}\sum_{m=-\infty}^{\infty}\int_{-1/2}^{1/2} |D_\delta(x+m)|\,dx = \frac{1}{2\delta}\int_{-\infty}^{\infty}|D_\delta(x)|\,dx = (4N+4)^{-1}.\end{aligned}$$

To obtain (7.8) we proceed in a manner quite similar to our proof of (2.16). From the Fourier expansion (2.20) we have

$$\int_{-1/2}^{1/2} p_N(x)\operatorname{sgn}\{\sin 2\pi(N+1)x\}\,dx = 0$$

for every trigonometric polynomial $p_N(x)$ having degree at most N. This leads to the lower bound

$$\text{(7.12)}\qquad \begin{aligned}&\int_{-1/2}^{1/2} |p_N(x) - \psi(x)|\,dx\\ &\geqslant \left|\int_{-1/2}^{1/2} \{p_N(x) - \psi(x)\}\operatorname{sgn}\{\sin 2\pi(N+1)x\}\,dx\right|\\ &= \left|\int_{-1/2}^{1/2} \psi(x)\operatorname{sgn}\{\sin 2\pi(N+1)x\}\,dx\right|\\ &= \left|\frac{2}{\pi i}\sum_{k=-\infty}^{\infty}(2k+1)^{-1}\int_{-1/2}^{1/2}\psi(x)e\{(2k+1)(N+1)x\}\,dx\right|\\ &= \pi^{-2}(N+1)^{-1}\sum_{k=-\infty}^{\infty}(2k+1)^{-2} = (4N+4)^{-1}.\end{aligned}$$

If there is equality in (7.12) then $p_N(x)$ interpolates the values of $\psi(x)$ at the points $l(2N+2)^{-1}$ for $l = 1, 2, \ldots, (2N+1)$. Since the degree of p_N is at most N, such an interpolating polynomial exists and is unique [**Zyg**, Vol. II, pp. 1–3]. In view of (7.6) the unique interpolating polynomial is precisely $\psi * i_N(x)$.

The determination of a unique extremal trigonometric polynomial in inequality (7.8) is a basic step in the sharp forms of Jackson's theorems obtained by Favard [**Fav**] and Achieser and Krein [**A-K**] (see also Cheney [**Che**, pp. 139–148]). A typical approximation theorem of this type is given below as (7.23).

Our next result identifies an extremal property associated with $\psi * j_N(x) + (2N+2)^{-1}k_N(x)$.

THEOREM 18. *The trigonometric polynomial*

$$\psi * j_N(x) = \sum_{\substack{n=-N \\ n\neq 0}}^{N} (-2\pi i n)^{-1}\hat{J}_{N+1}(n)e(nx)$$

satisfies

$$\operatorname{sgn}(\psi * j_N(x)) = \operatorname{sgn}(\psi(x)), \tag{7.13}$$

$$|\psi * j_N(x) - \psi(x)| \leqslant (2N+2)^{-1}k_N(x), \tag{7.14}$$

and

$$|\psi * j_N(x)| \leqslant |\psi(x)|. \tag{7.15}$$

If $p_N(x)$ is any trigonometric polynomial of degree at most N satisfying $p_N(x) \geqslant \psi(x)$ for all real x, then

$$\int_{-1/2}^{1/2} p_N(x) - \psi(x)\,dx \geqslant (2N+2)^{-1}. \tag{7.16}$$

Moreover, there is equality in (7.16) *if and only if*

$$p_N(x) = \psi * j_N(x) + (2N+1)^{-1}k_N(x).$$

PROOF. Identity (7.13) is established in the same way as (7.5).

Next we let $E(x)$ be defined as in Corollary 7. Mimicking our proof of (7.10) we find that

$$(2N+2)^{-1}\sum_{m=-\infty}^{\infty} E_{N+1}(x+m) = \psi * j_N(x) - \psi(x). \tag{7.17}$$

Hence,

$$|\psi * j_N(x) - \psi(x)| \leqslant (2N+2)^{-1}\sum_{m=-\infty}^{\infty} K_{N+1}(x+m) = (2N+2)^{-1}k_N(x)$$

by combining (2.24), (7.3), and (7.17).

Inequality (7.15) is somewhat more obscure. We use the identity (2.26) with $x > 0$. A simple calculation shows that

$$\begin{aligned} -E(x) = 1 - H(x) &= \left(\frac{\sin \pi x}{\pi}\right)^2\left\{-2x^{-1} + x^{-2} + 2\sum_{m=1}^{\infty}(x+m)^{-2}\right\} \\ &= \left(\frac{\sin \pi x}{\pi}\right)^2\left\{\int_0^{\infty}\left(-2 + u + 2\sum_{m=1}^{\infty} ue^{-um}\right)e^{-ux}\,du\right\} \\ &= \left(\frac{\sin \pi x}{\pi}\right)^2\left\{4\int_0^{\infty}(v\coth v - 1)e^{-2vx}\,dv\right\}. \end{aligned}$$

Since $v\coth v > 1$ for $v > 0$ we see that

$$-E(x) = ((\sin \pi x)/\pi)^2 R(x) \tag{7.18}$$

for $x > 0$, where $R(x)$ is a positive, strictly decreasing function of x. Now we use (7.17), (7.18), and the fact that $E(x)$ is an odd function. It follows that

$$(7.19)\quad \psi * j_N(x) - \psi(x)$$
$$= -(2N+2)^{-1}\left\{\sum_{m=0}^{\infty} E_{N+1}(x+m) - \sum_{m=0}^{\infty} E_{N+1}((1-x)+m)\right\}$$
$$= (2N+2)^{-1}\left(\frac{\sin \pi(N+1)x}{\pi}\right)^2$$
$$\cdot \sum_{m=0}^{\infty} \{R_{N+1}(x+m) - R_{N+1}((1-x)+m)\}.$$

From our observations concerning $R(x)$, the right side of (7.19) is clearly positive for $0 < x < 1/2$. This, together with (7.13), shows that (7.15) holds for $0 < x < 1/2$. Then the inequality holds for all real x, since both sides of (7.15) are even functions of period 1.

Finally, we suppose that $p_N(x)$ has degree at most N and majorizes $\psi(x)$. By continuity we must have $p_N(0) \geqslant 1/2$. Thus,

$$(7.20)\quad \frac{1}{2} \leqslant \sum_{l=0}^{N}\left\{p_N\left(l(N+1)^{-1}\right) - \psi\left(l(N+1)^{-1}\right)\right\}$$
$$= \sum_{n=-N}^{N} \hat{p}_N(n) \sum_{l=0}^{N} e\left(ln(N+1)^{-1}\right) - \sum_{l=1}^{N}\left\{l(N+1)^{-1} - \frac{1}{2}\right\}$$
$$= (N+1)\hat{p}_N(0).$$

As $\hat{\psi}(0) = 0$, (7.20) is equivalent to (7.16). If (7.20) holds with equality then

$$(7.21)\quad p_N(0) = 1/2 \quad \text{and} \quad p_N\left(l(N+1)^{-1}\right) = l(N+1)^{-1} - 1/2$$

for $l = 1, 2, \ldots, N$. Since $p_N(x)$ also majorizes $\psi(x)$, we must have

$$(7.22)\qquad p_N'\left(l(N+1)^{-1}\right) = 1$$

for $l = 1, 2, \ldots, N$. It follows (see Zygmund [**Zyg**, Vol. II, p. 23]) that the $2N+1$ conditions (7.21) and (7.22) determine a unique polynomial of degree at most N. From (7.13)–(7.15) we see that these conditions are also satisfied by $\psi * j_N(x) + (2N+2)^{-1}k_N(x)$. This completes our proof of the theorem.

We now state a result which is the periodic analogue of Theorem 11. Here we suppose that $f\colon \mathbf{R} \to \mathbf{C}$ has period 1 and bounded variation on each closed interval of length 1. We also assume that f satisfies the normalizing condition (1.16). The total variation of f on $[-\frac{1}{2}, x]$ will be denoted by $V_f(x)$, and $V_f = V_f(\frac{1}{2})$. We write $(dV_f) * k_N(x)$ for the convolution

$$(dV_f) * k_N(x) = \int_{-1/2}^{1/2} k_N(x-\xi)\, dV_f(\xi).$$

THEOREM 19. *The trigonometric polynomials $f * i_N(x)$, $f * j_N(x)$, and $(dV_f) * k_N(x)$ have degree at most N and satisfy*

$$(7.23)\qquad \int_{-1/2}^{1/2} |f(x) - f * i_N(x)|\, dx \leqslant (4N+4)^{-1} V_f,$$

and

$$(7.24)\qquad |f(x) - f * j_N(x)| \leqslant (2N+2)^{-1}(dV_f) * k_N(x).$$

In particular, if f is real valued, then

$$(7.25)\qquad m(f, N, x) = f * j_N(x) + (2N+2)^{-1}(dV_f) * k_N(x)$$

has degree at most N, satisfies

$$(7.26)\qquad f(x) \leqslant m(f, N, x),$$

and

$$(7.27)\qquad \int_{-1/2}^{1/2} m(f, N, x) - f(x)\, dx = (2N+2)^{-1} V_f.$$

PROOF. From (7.4) we have

$$(d/dx)\psi * i_N(x) = 1 - i_N(x).$$

This can be used to verify the identity

$$\begin{aligned}(7.28)\quad \int_{-1/2}^{1/2} & f(x-\xi)\, d\{\psi * i_N(\xi) - \psi(\xi)\} \\ &= \int_{-1/2}^{1/2} f(x-\xi)\{1 - i_N(\xi)\}\, d\xi - \int_{-1/2}^{1/2} f(x-\xi)\, d\psi(\xi) \\ &= f(x) - f * i_N(x)\end{aligned}$$

at all continuity points x of f. Integrating the left side of (7.28) by parts, we find that

$$(7.29)\quad \int_{-1/2}^{1/2} \psi * i_N(x-\xi) - \psi(x-\xi)\, df(\xi) = f(x) - f * i_N(x)$$

at all continuity points x. Since f is continuous almost everywhere,

$$\begin{aligned}\int_{-1/2}^{1/2} & |f(x) - f * i_N(x)|\, dx \\ &\leqslant \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |\psi * i_N(x-\xi) - \psi(x-\xi)|\, dx\, dV_f(\xi) \\ &= (4N+4)^{-1} V_f.\end{aligned}$$

Similarly, we deduce (7.29), but with i_N replaced by j_N. Hence, at each continuity point x of f we have

$$\begin{aligned}(7.30)\quad |f(x) - f * j_N(x)| &\leqslant \int_{-1/2}^{1/2} |\psi * j_N(x-\xi) - \psi(x-\xi)|\, dV_f(\xi) \\ &\leqslant (2N+2)^{-1}\int_{-1/2}^{1/2} k_N(x-\xi)\, dV_f(\xi) \\ &= (2N+2)^{-1}(dV_f) * k_N(x).\end{aligned}$$

Using the normalization (1.9) we find that (7.24) holds for all real x. Of course, (7.26) is an immediate consequence of (7.30).

8. The Erdös-Turán inequality. Let E denote an interval of $\mathbf{R}$ having length $|E| < 1$ and let $\chi_E(x)$ be the normalized characteristic function of E. Throughout this section $\sigma_E(x)$ will denote the periodic function

$$\sigma_E(x) = \left\{ \sum_{l=-\infty}^{\infty} \chi_E(x+l) \right\} - |E|. \tag{8.1}$$

We also suppose that $\{\xi_1, \xi_2, \ldots\}$ is a sequence of real numbers, and we write $\vec{\xi}_M = \{\xi_1, \xi_2, \ldots, \xi_M\}$ for the first M terms. As is well known, the sequence $\{\xi_1, \xi_2, \ldots\}$ is said to be uniformly distributed mod 1 if

$$\lim_{M\to\infty} M^{-1} \sum_{m=1}^{M} \sigma_E(\xi_m) = 0 \tag{8.2}$$

for every interval E. By Weyl's criterion [**K-N**, p. 7] this is equivalent to the requirement that

$$\lim_{M\to\infty} M^{-1} \sum_{m=1}^{M} e(l\xi_m) = 0$$

for every positive integer l.

For each positive integer M we define the discrepancy $\Delta(\vec{\xi}_M)$ of $\vec{\xi}_M$ by

$$\Delta(\vec{\xi}_M) = \sup_E M^{-1} \left| \sum_{m=1}^{M} \sigma_E(\xi_m) \right|.$$

Then another form of Weyl's criterion states that the sequence $\{\xi_1, \xi_2, \ldots\}$ is uniformly distributed mod 1 if and only if $\lim_{M\to\infty} \Delta(\vec{\xi}_M) = 0$. An important inequality of Erdös and Turán [**E-T**] provides an upper bound for the discrepancy $\Delta(\vec{\xi}_M)$ in terms of the sums $M^{-1}\sum_{m=1}^{M} e(l\xi_m)$ for l in a finite range (see also [**N-P**] and [**K-N**, p. 112]). Our ability to majorize $\sigma_E(x)$ by a trigonometric polynomial is ideally suited to proving results of this type. In fact, we could also use (8.1) and Selberg's function (1.5), which majorizes $\chi_E(x)$. This application, as well as our Corollary 21, was first observed by H. L. Montgomery.

THEOREM 20. *For each interval E of length $|E| < 1$ and every integer $N \geqslant 1$ we have*

$$\left| \sum_{m=1}^{M} \sigma_E(\xi_m) \right| \leqslant (N+1)^{-1} \left\{ M + 2 \sum_{n=1}^{N} \hat{K}_{N+1}(n) \left| \sum_{m=1}^{M} e(n\xi_m) \right| \right\} + 2 \sum_{n=1}^{N} \frac{|\sin \pi n|E||}{\pi n} \hat{J}_{N+1}(n) \left| \sum_{m=1}^{M} e(n\xi_m) \right|. \tag{8.3}$$

PROOF. If $f(x)$ satisfies the hypotheses of Theorem 19 and $\hat{f}(0) = 0$, then

$$\begin{aligned}(8.4)\quad \left|\sum_{m=1}^{M} f(\xi_m)\right| &\leqslant (2N+2)^{-1} \sum_{m=1}^{M} (dV_f) * k_N(\xi_m) + \left|\sum_{m=1}^{M} f * j_N(\xi_m)\right| \\ &\leqslant (2N+2)^{-1} V_f \sum_{n=-N}^{N} \hat{K}_{N+1}(n) \left|\sum_{m=1}^{M} e(n\xi_m)\right| \\ &\quad + \sum_{\substack{n=-N \\ n\neq 0}}^{N} |\hat{f}(n)| \hat{J}_{N+1}(n) \left|\sum_{m=1}^{M} e(n\xi_m)\right|.\end{aligned}$$

Now we let $f(x) = \sigma_E(x)$. It follows that $V_f = 2$ and

$$|\hat{f}(n)| = (\pi n)^{-1} |\sin \pi n\, |E||$$

for $n \neq 0$. Hence, inequality (8.4) implies (8.3).

If we ignore the factors $\hat{K}_{N+1}$ and $\hat{J}_{N+1}$ on the right of (8.3), we obtain a bound on the discrepancy of the form

$$\Delta(\vec{\xi}_M) \leqslant (N+1)^{-1} + 2 \sum_{n=1}^{N} \left\{(\pi n)^{-1} + (N+1)^{-1}\right\} \left| M^{-1} \sum_{m=1}^{M} e(n\xi_m)\right|.$$

Alternatively, if the length $|E|$ is small, the factor $|\sin \pi n |E||$ can be used to advantage.

COROLLARY 21. *Suppose that $\xi_1, \xi_2, \ldots, \xi_M$ are real numbers satisfying $\|\xi_m\| \geqslant X^{-1}$ for some real parameter $X \geqslant 2$. Then*

$$(8.5)\qquad M \leqslant 6 \sum_{1 \leqslant n \leqslant X} \left|\sum_{m=1}^{M} e(n\xi_m)\right|.$$

PROOF. Let $E = (-X^{-1}, X^{-1})$ and $N = [X]$. Since the right side of (8.5) is continuous in each ξ_m, we may suppose $\|\xi_m\| > X^{-1}$. Then

$$(8.6)\qquad \left|\sum_{m=1}^{M} \sigma_E(\xi_m)\right| = M|E| = 2MX^{-1}$$

and, using $|\sin \pi n|E|| \leqslant \pi n|E|$, this is bounded from above by

$$\begin{aligned}(8.7)\quad &(N+1)^{-1}M + 2\sum_{n=1}^{N} \left\{(N+1)^{-1} + |E|\right\} \left|\sum_{m=1}^{M} e(n\xi_m)\right| \\ &\leqslant MX^{-1} + 6X^{-1} \sum_{n=1}^{N} \left|\sum_{m=1}^{M} e(n\xi_m)\right|.\end{aligned}$$

The corollary follows from (8.6) and (8.7).

We note that (8.5) has been used recently in the work of Baker and Harman **[B-H]**.

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