RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

CANONICAL PERTURBATION THEORY OF ANOSOV SYSTEMS, AND REGULARITY RESULTS FOR THE LIVSIC COHOMOLOGY EQUATION

BY R. DE LA LLAVE, J. M. MARCO AND R. MORIYÓN

We have studied the following problem.

Given a compact symplectic manifold (M, ω) and a C^∞ family of C^∞ symplectic transformations f_ε, when is it true that there exists another family of C^∞ diffeomorphisms g_ε such that

$$(*) \qquad g_\varepsilon \circ f_0 = f_\varepsilon \circ g_\varepsilon?$$

The motivation, of course, is canonical perturbation theory, where f_0 is supposed to be "well understood", and we want to use this knowledge to get information about similar systems. Typically, f_0 is an integrable system, but there are nonintegrable systems whose behaviour is also known in great detail, and one could consider perturbation theories for them. Indeed, particular perturbation theories have been considered for geodesic flows in surfaces or manifolds of negative curvature [**G.K.2.3**; **C.E.G**].

Our results show that there is a very satisfactory answer when f_0—and hence f_ε for small ε—have some hyperbolicity in them.

DEFINITION. We say f_ε is a globally hamiltonian isotopy (G.H.I.) when

$$df_\varepsilon/d\varepsilon = \tilde{F}_\varepsilon \circ f_\varepsilon,$$

$\tilde{F}_\varepsilon$ being a globally hamiltonian vector field of hamiltonian F_ε with vanishing average (similarly for local hamiltonian isotopies, L.H.I.).

Since F_ε determines f_ε, if we only look for g_ε which are G.H.I. starting at the identity, $(*)$ is equivalent to

$$(**) \qquad F_\varepsilon = G_\varepsilon - G_\varepsilon \circ f_\varepsilon^{-1},$$

This is roughly equivalent to the use of generating functions in classical theory to rewrite equations between diffeomorphisms as equations between functions. Generating functions cannot be defined in all manifolds, whereas

Received by the editors August 2, 1984 and, in revised form, September 11, 1984.
1980 *Mathematics Subject Classification*. Primary 58F05, 58F15, 70H15.

G.H.I. can. Whenever generating functions can be defined, it can be shown that a family admitting them is a G.H.I., and vice-versa under smallness conditions which are natural for perturbations, so the theory is strictly more general.

Notice also that $(**)$ is a linear cohomology equation, so we believe that this formalism should also be useful in many other computations.

For example, Nash-Moser techniques would allow finding solutions whenever we can solve—e.g., by Fourier analysis—the equation for $\varepsilon = 0$ with tame estimates.

When there is enough hyperbolicity, we can solve $(**)$ for any ε and prove regularity of the solutions, improving the classical Livsic result [**L1**]. From this we are able to solve $(*)$ directly. The relevant results are the following. (See [**L1**; **L2**; **G.K.1**], for previous results in this direction).

THEOREM 1. *Let f be a transitive C^∞ Anosov diffeomorphism on a compact manifold M, and η a C^∞ function satisyfing $\sum_{k=0}^{N-1} \eta(f^k x) = 0$ for all points x such that $f^N(x) = x$. Then there exists a C^∞ function ψ such that $\psi - \psi \circ f = \eta$. If both f and η vary in a C^∞ fashion with a parameter $\varepsilon \in \mathbb{R}^n$, so does ψ, assuming that $\psi_\varepsilon(x_\varepsilon)$ is C^∞ for some C^∞ curve x_ε.*

SKETCH OF THE PROOF. Since

$$\psi(f^k x) = \psi(x) - \sum_{j=0}^{k-1} \eta(f^j x),$$

an explicit formula can be given for the value of ψ at points in the stable manifold of a periodic point in terms of its value at the periodic point and the values of η at the orbit of the initial point. This allows the computation of derivatives of ψ of arbitrary order along stable directions. A similar argument shows that ψ is also of class C^∞ along any unstable manifold.

If the stable and unstable foliations were smooth, we could form elliptic operators with derivatives along stable and unstable directions, and the smoothness of ψ would follow from the elliptic theory of linear differential operators. In the general case the problem of the regularity of ψ can be reduced to a question about the domains of definition of elliptic linear differential operators with Hölder continuous coefficients. We do not know how to solve this problem in general, but an improvement of Anosov's argument proving absolute continuity of the stable and unstable foliations (see [**A**]) reduces our question to the case of a positive symmetric operator. In this case Schauder theory proves the regularity of ψ.

The C^∞ dependence with respect to parameters can be proved by formal differentiation of the given equation, giving rise to a new equation of the same type, whose solution can be integrated. In this process it is essential to show that equations like $\psi_\varepsilon - \psi_\varepsilon \circ f_\varepsilon = \eta_\varepsilon$ can be written as $\tilde{\psi}_\varepsilon - \tilde{\psi}_\varepsilon \circ f_0 = \tilde{\eta}_\varepsilon$ with some regularity with respect to the parameter. This is a consequence of Moser's proof of the structural stability of Anosov diffeomorphisms [**S**].

As a corollary we have

THEOREM 2. *let f be as above. Assume there is a continuous invariant measure of density $\rho > 0$. Then ρ is C^∞.*

Using Theorem 1 we can prove

THEOREM 3. *Let f_ε be a G.H.I., each f_ε being a transitive Anosov diffeomorphism on a compact manifold. The following are equivalent*:

(a) $\exists g_\varepsilon \in$ *G.H.I. such that* $f_\varepsilon \circ g_\varepsilon = g_\varepsilon \circ f_0$.

(b) $\forall x \in \mathrm{Per}_N(f_\varepsilon), N^{-1}\sum_{i=0}^{N-1} F_\varepsilon(f_\varepsilon^i x) = 0$.

Moreover, under the additional assumption that $(I - f)_{0\#}$ is an isomorphism in the first cohomology class, these are equivalent to

(c) $\exists g_\varepsilon \in$ *L.H.I. such that* $f_\varepsilon \circ g_\varepsilon = g_\varepsilon \circ f_0$. *Moreover, in this case, any* g_ε *as above is a G.H.I.*

If we assume that (M, ω) admits a prequantization [**K**], then action invariants of periodic orbits can be defined. These invariants are independent of ε if and only if (a) or (b) holds.

It is a well-known conjecture [**B**] that the assumption before (c) is always true. If this conjecture were right, the problem of conjugacy under L.H.I. would have a very neat answer because we have:

THEOREM 4. (a) *Assume* $f_\varepsilon, g_\varepsilon \in$ *L.H.I. and* $f_\varepsilon \circ g_\varepsilon = g_\varepsilon \circ f_0$. *Then the cohomology class of* $i(\tilde{F}_\varepsilon)\omega$ *is in the image of* $(I - f_{0\#})$. (b) *Conversely, assume that the class of* $i(\tilde{F}_\varepsilon)\omega$ *is in the image of* $(I - f_{0\#})$. *Then there exists* $h_\varepsilon \in$ *L.H.I. such that* $(h_\varepsilon^{-1} \circ f_\varepsilon \circ h_\varepsilon) \in$ *G.H.I.*

An analogue of Theorems 3 and 4 can be developed for hamiltonian flows that are Anosov on each energy surface. Its proof is based on an analogue of Theorem 1 for flows, so that, under the compatability conditions, we can solve the cohomology equation on each energy surface, and the differentiability with respect to the energy is proved, considering it as a parameter. The proof of the analogue of Theorem 1 for flows is very similar to that for diffeomorphisms. We need differentiability of the conjugating homeomorphism with respect to parameters in the Anosov theorem. This can be achieved by giving a proof of the Anosov theorem by the implicit function theorem along the lines of the proof of J. Moser [**M**].

Therefore, the methods can be adapted to discuss conjugations by G.H.I., and further modifications also allow the inclusion of time changes for each energy surface.

For both types of conjugation we compute necessary and sufficient conditions and action invariants as in the case of diffeomorphisms.

For analytic families of analytic diffeomorphisms, it is also possible to develop an asymptotic perturbation theory. It can furthermore be shown that, if the eliminations can be carried out to all orders in ε, then Theorem 2(b) holds, and hence there is a C^∞ g_ε solving $(*)$.

Detailed proofs of the above results will appear elsewhere.

REFERENCES

[A] D. V. Anosov, *Geodesic flows and closed riemannian manifolds with negative curvature,* Proc. Steklov Inst. Math. **90** (1967).

[B] R. Bowen, *On Axiom A diffeomorphisms,* CBMS Regional Conf. Ser. in Math., no. 35, Amer. Math. Soc., Providence, R.I., 1978.

[C.E.G.] P. Collet, H. Epstein and G. Gallavotti, *Perturbations of geodesic flows on surfaces of constant negative curvature and their mixing properties,* I.M.A. preprint 23, Comm. Math. Phys. (to appear).

[G.K.1] V. Guillemin and I. Kazhdan, *On the cohomology of certain dynamical systems,* Topology **19** (1980), 291–299.

[G.K.2] ——, *Some inverse spectral results for negatively curved 2-manifolds,* Topology **19** (1980), 301–312.

[G.K.3] ——, *Some inverse spectral results for negatively curved n-manifolds,* Geometry of the Laplace Operator, Proc. Sympos. Pure Math., vol. 36, Amer. Math. Soc., Providence, R.I., 1980, pp. 153–180.

[K] B. Konstant, *Quantization and unitary representations,* Lectures in Modern Analysis, III (C. T. Taam, ed.), Lecture Notes in Math., vol. 170, Springer-Verlag, 1970, pp. 87–207.

[L1] A. L. Livsic, *Cohomology properties of Y-systems,* Math. Notes **10** (1971), 758–763.

[L2] ——, *Cohomology of dynamical systems,* Math. USSR-Izv. **6** (1972), 1278–1301.

[M] J. Moser, *On a theorem of Anosov,* J. Differential Equations **5** (1969), 411–440.

[S] S. Smale, *Differentiable dynamical systems,* Bull. Amer. Math. Soc. **73** (1967), 747–817.

I.H.E.S., 91440 BURES SUR YVETTE, FRANCE

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON NEW JERSEY 08544 (Current address of R. de la Llave)

MATEMATICAS, UNIVERSIDAD AUTONOMA DE MADRID, 28049 MADRID, SPAIN (Current address of J. M. Marco and R. Mariyón)