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ON WHITEHEAD'S ALGORITHM

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ABSTRACT. One can decide effectively when two finitely generated subgroups of a finitely generated free group F are equivalent under an automorphism of F. The subgroup of automorphisms of F mapping a given finitely generated subgroup S of F into a conjugate of S is finitely presented.

In two famous articles [**9**, **10**] which appeared in 1936, J. H. C. Whitehead, using the theory of three-dimensional handlebodies, proved that one can effectively decide when two n-tuples of cyclic words of a finitely generated free group F are equivalent by an automorphism of F. The proof of this result has been simplified successively [**7**, **3**] and the result itself has been immensely influential. Whitehead himself poses the problem of generalizing his theorem [**10**, p. 800]; namely he raises the question of deciding when two finitely generated subgroups of F are equivalent by an automorphism of F.

In 1974 McCool [**6**] deduced a profound consequence of Whitehead's theorem, proving that the stabilizer, in the automorphism group of F, of an n-tuple of cyclic words is finitely presented. Using graph-theoretic techniques we developed in [**1**] (the results of which were announced in [**2**]), we have succeeded both in settling Whitehead's question and in generalizing McCool's results.

Let A denote the automorphism group of F, and let $\mathcal{S}$ denote the set of conjugacy classes of finitely generated subgroups of F with its natural A action. Let $\mathcal{S}^n$ denote the cartesian product of n copies of $\mathcal{S}$ with diagonal A action.

THEOREM W. *There is an effective procedure for determining when two elements of $\mathcal{S}^n$ are in the same orbit of the A-action.*

THEOREM M. *The stabilizer in A of an element of $\mathcal{S}^n$ is finitely presented, and a finite presentation can be effectively determined.*

In this note we indicate briefly the ideas that go into the proofs of Theorems W and M. Full details will appear elsewhere.

We use the theory of *graphs* defined in [**2**]. A graph X is a nonempty set with involution, denoted $x \mapsto \bar{x}$, together with a retraction $\iota: X \to V(X)$ of X onto the fixed point set $V(X)$ of the involution. Morphisms of graphs preserve the involution and the retraction. The set $V(X)$ is called the set of *vertices* of

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X and $E(X) = X - V(X)$ is called the set of *edges*. A morphism $f: X \to X'$ of graphs is called an *immersion* [8] if for each $v \in V(X)$ the induced map $f_v: \mathrm{Star}_X(v) \to \mathrm{Star}_{X'}(f(v))$ is injective; here $\mathrm{Star}_X(v) = \{x \in X \mid \iota x = v\}$. A graph X is called a *core graph* if it has no end vertices, where $v \in V(X)$ is called an *end vertex* if there exists precisely one edge e with $\iota e = v$. *The graphs considered in this note* (except for coverings in parenthetical remarks) *are all finite.*

Suppose now that $j: X \to Y$ is an immersion of the core graph X in the 1-vertex graph Y (i.e. $\#V(Y) = 1$). Define the *complexity* of j, or of X by abuse of notation, to be $\#V(X)$ and denote it by $c(X)$. The crucial algebraic result below will enable us to compute the effect of a Whitehead automorphism of $\pi_1(Y)$ [4, p. 31] on the complexity $c(X)$. (That $A = \mathrm{Aut}\,\pi_1(Y)$ acts on immersions $X \xrightarrow{j} Y$ may be seen as follows. If X is connected, and $v \in V(X)$, then j injects $\pi_1(X, v)$ into $\pi_1(Y)$ to determine a conjugacy class of subgroups of $\pi_1(Y)$. If $\alpha \in A$, represent the subgroup $\alpha(j(\pi_1(X, v)))$ of $\pi_1(Y)$ as a covering of Y and take a core of the covering to get the desired immersion $\alpha(X) \xrightarrow{\alpha(j)} Y$.)

If $A, B \subseteq E(Y)$ and $v \in V(X)$, define $(A \cdot B)_v$ to be 1 if there exists a reduced path ee' in X (so $e, e' \in E(X)$) with $\iota\bar{e} = \iota e' = v$, $je \in A$, and $j(\overline{e'}) \in B$, and let $(A \cdot B)_v$ be 0 otherwise. Set $A \cdot B = \sum_{v \in V(X)} (A \cdot B)_v$. Thus $A \cdot B$ is the number of vertices v of X for which a reduced path ee' exists in X with $j(e) \in A$, $j(\overline{e'}) \in B$, and $\iota(\bar{e}) = \iota(e') = v$.

PROPOSITION 1. *If $\alpha = (A, a)$ is a Whitehead automorphism of $\pi_1(Y)$ ($A \subset E(Y)$, $a \in A$, $\bar{a} \notin A$) and $X \xrightarrow{j} Y$ is an immersion of the core graph X in the one vertex graph Y, then $c(\alpha(X)) - c(X) = A \cdot A' - \{a\} \cdot E(Y)$. Here $A' = E(Y) - A$.*

This result reduces to Proposition 4.16, p. 31 of [4] in the special case when X is the graph whose geometric realization is a subdivision of the circle. The formal properties of the pairing $A \cdot B$ are:

(1) $A \cdot B = B \cdot A \geq 0$;

(2) $\{a\} \cdot \{a\} = 0$ if $a \in E(Y)$,

(3) $\{a\} \cdot E(Y) = \#\{v \in V(X) \mid \exists e \in \mathrm{Star}_X(v) \text{ with } f(e) = a\} = \{\bar{a}\} \cdot E(Y)$, if $a \in E(Y)$.

The pairing $A \cdot B$ is not bilinear over disjoint unions, unlike the special case considered in [4, p. 31]. However a weaker result holds.

PROPOSITION 2. *For any subsets A, B of $E(Y)$ one has*

$$A \cdot A' + B \cdot B' \geq (A \cap B) \cdot (A \cap B)' + (A' \cap B') \cdot (A' \cap B')'.$$

In fact the analogous inequality holds locally at each vertex of X.

PROPOSITION 3. *Let A and B be subsets of $E(Y)$ with $A \cap B = \varnothing$, $a \in A$, $\bar{a} \notin A$, $b \in B$, $\bar{b} \notin B$, and $\bar{a} \notin B$. Let $\sigma = (A, a)$ and $\tau = (B, b)$ be Whitehead automorphisms of $\pi_1(Y)$. Then for any immersion $j: X \to Y$ of the core graph X into the 1-vertex graph Y, one has*

$$c(\tau\sigma(X)) - c(\sigma X) = c(\tau X) - c(X).$$

Using Propositions 1–3 and following the plan of the argument of Lemma 4.18 of [4], one proves

THEOREM 1. *Suppose* $j: X \to Y$ *is an immersion, where* X *is a core graph and* Y *is a 1-vertex graph. Let* σ *and* τ *be Whitehead automorphisms of* $\pi_1(Y)$ *such that* $c(\sigma(X)) \leq c(X)$ *and* $c(\tau(X)) \leq c(X)$, *where at least one inequality is strict. Then using only McCool's relations* R1–R7 [5] *one has* $\tau\sigma^{-1} = \sigma_m \cdots \sigma_2\sigma_1$, *where* σ_i *are Whitehead automorphisms and where* $c(\sigma_i \cdots \sigma_1\sigma(X)) < c(X)$ *for* $1 \leq i < m$.

Suppose now that S is a conjugacy class of finitely generated subgroups of $\pi_1(Y)$. Then S determines (by taking a covering of Y and taking a core of the cover) an immersion $j: X \to Y$ of a finite core graph X in Y such that $j(\pi_1(X, v))$ is in the conjugacy class S; the graph X is unique up to isomorphism, so we may define the complexity

$$c(S) = c(X) = \#V(X).$$

Observe that in the special case where S is represented by the cyclic group $\langle w \rangle$, $c(S)$ is just the length of a cyclically reduced word conjugate to w. Observe also that if some representative of S has finite index n in $\pi_1(Y)$ (whence all representatives have index n) then $c(S) = n$, since the immersion corresponding to S is an n-fold covering space of Y in this case.

COROLLARY 1. *Let* F *be a finitely generately free group with given free basis* $\mathcal{O}$ *and let* $S = (S_1, S_2, \dots, S_n)$ *be an* n*-tuple of conjugacy classes of finitely generated subgroups of* F. *Let* $c(S) = \sum_{i=1}^{n} c(S_i)$. *Suppose that* σ *and* τ *are Whitehead automorphisms of* F *such that* $c(\sigma(S)) \leq c(S)$ *and* $c(\tau(S)) \leq c(S)$ *with at least one inequality strict. Then using only McCool's relations* R1–R7 *one has* $\tau\sigma^{-1} = \sigma_m \cdots \sigma_2\sigma_1$, *where* σ_i *are Whitehead automorphisms and where* $c(\sigma_i \cdots \sigma_1\sigma(S)) < c(S)$ *for* $1 \leq i < m$.

An immediate consequence of Corollary 1 is

COROLLARY 2. *If* $c(S)$ *can be reduced by some automorphism of* F, *then it can be reduced by a Whitehead automorphism.*

Theorem W follows from Corollary 2 by the method of proof of Proposition 4.19 of [4].

We remark that Theorem M follows from Corollary 1 by arguments mimicking McCool's [6].

EXAMPLE. Suppose S is a finitely generated subgroup of F whose conjugacy class has complexity 1. Then using Stallings' form of Marshall Hall's theorem [8] it follows that S is a free factor of F. Whitehead found another algorithm to detect when S is a free factor of F, based on the existence of a cut vertex in the (based) star graph of a basis for S [9]. We have also given a direct proof of this result using our graph techniques, avoiding any use of handlebodies.

REMARK. The novel feature of our work is our definition of the complexity $c(S)$ of an n-tuple S of conjugacy classes of finitely generated subgroups of a free group. Whitehead's own example [10, p. 800], indicating the difficulty of his problem of deciding when two fg subgroups of the free group F were equivalent, when reexamined in this light, shows that he was working with the wrong notion of complexity of a subgroup (he uses the sum of lengths of the elements of a given free basis for a subgroup). It is our complexity, defined in terms of the core of a covering space, which satisfies the correct transformation formula under Whitehead automorphisms, so that Whitehead's own arguments will work. Whitehead's examples [**10**, p. 800], $S = \langle (a\bar{b})^2\bar{b}^2(a\bar{b})^2a^3, \bar{a}^3\bar{b}^5\rangle$, $T = \langle a^2\bar{b}^2a^2\bar{b}^5, (ab)^{-3}\bar{b}^5\rangle$, subgroups of $F(a,b)$, have complexities 17 and 16 respectively (but lengths 21 and 22 repectively). They are equivalent by the Whitehead map $(\{a,b\},b)$.

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