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PROPER HOLOMORPHIC MAPPINGS

BY ERIC BEDFORD[1]

Contents

Introduction. Let us recall that a mapping $F: X \to Y$ is *proper* if $f^{-1}(K)$ is a compact subset of X whenever $K \subset Y$ is compact. If X and Y are complex spaces, and if $F: X \to Y$ is a proper holomorphic mapping, then $F^{-(}y_0)$ is a compact analytic subvariety of X for all points $y_0 \in Y$. Proper mappings between complex spaces were studied from the general point of view of complex spaces in the 1950s and early 60s (see Remmert-Stein [**78**]). Two results from this era are a factorization theorem of Stein [**88**] and the Remmert Proper Mapping theorem: *If $f: X \to Y$ is a proper mapping, and if $S \subset X$ is a subvariety of X, then $f(S)$ is a subvariety of Y.*

Here we consider a special case: proper mappings $F: \Omega \to D$ where $\Omega \subset\subset X = \mathbf{C}^n$ and $D \subset\subset Y = \mathbf{C}^N$ are smoothly bounded domains.[2] The letters Ω and D will always denote domains of $\mathbf{C}^n$, and a "proper mapping" will always be assumed to be holomorphic. (In many cases the same results are valid in the case where X and Y are Stein manifolds, although we will not emphasize this point.)

It is evident that a mapping $F: \Omega \to D$ is proper if and only if f maps $\partial\Omega$ to ∂D in the following sense:

$$\text{if } \{z_j\} \subset \Omega \text{ is a sequence with } \lim_{j\to\infty} \operatorname{dist}(z_j, \partial\Omega) = 0, \text{ then } \lim_{j\to\infty} \operatorname{dist}(f(z_j), \partial D) = 0.$$

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[2] In this case proper mappings are also known as "finite mappings".

Thus proper mappings $f\colon \Omega \to D$ lead naturally to the geometric function theory of mappings taking $\partial\Omega$ to ∂D.

The case where proper mappings are best understood is when the domains are strongly pseudoconvex. Pinčuk **[73]** has shown that a proper mapping $f\colon \Omega \to D$ between strongly pseudoconvex domains is locally biholomorphic. In fact, this result remains true if D is only assumed to be (weakly) pseudoconvex (see **[20, 38]**). It has been shown, too, that if $\partial\Omega$ and ∂D are strongly pseudoconvex and real analytic then a germ of a holomorphic mapping f with $f(U \cap \partial\Omega) \subset \partial D$ for some open U containing $z_0 \in \partial\Omega$ may be analytically continued along $\partial\Omega$ (see **[71, 72, 27]**).

In this article we will survey some recent results on proper mappings between smoothly bounded domains. Biholomorphic mappings, of course, are a special case of proper ones. But here we want to study the features of proper maps that do not arise already in the locally biholomorphic case. Thus we focus our attention on the behavior of singular, i.e. branched, mappings, with emphasis on the smoothness and branching behavior at the boundary.

For boundary smoothness of proper mappings, we will present one of the proofs of S. Bell. We will not attempt to discuss or even mention the numerous other methods that have been developed; the reader may consult the references given in **[26, 37, 43]**. The literature on biholomorphic maps and automorphisms is so extensive that we cannot adequately present it here and, therefore, have omitted it almost entirely.

A related topic, not discussed here, is the proper mapping of polyhedra (see **[57, 78, 79, 80, 50, 85]**). Another topic which we have omitted is the holomorphic correspondence, i.e. multiple-valued analytic mapping. This arises naturally in problems where it is desirable to study f^{-1}, for example, in factorization. The reader is referred to **[89, 86, 80, 13, 14, 97]**.

Finally we note that, except in §6, we assume f is equidimensional, i.e. $\dim \Omega = \dim D$.

1. Structure and examples. First we give some examples of proper mappings, then we describe the basic structure of a proper map in terms of a branched cover, and we conclude this section with some more examples.

EXAMPLE (UNIT DISK). The proper mappings $f\colon \Delta \to \Delta$ are finite Blaschke products, i.e.

$$f(z) = e^{i\phi} \prod_{j=1}^{k} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z},$$

where $\alpha_1, \ldots, \alpha_k \in \Delta$, and $\phi \in \mathbf{R}$.

Proper mappings of the polydisk $\Delta^n = \Delta \times \cdots \times \Delta$ to itself are thus determined by the following theorem (see **[78, 69, 80]**).

THEOREM. *If $\Omega_1, \ldots, \Omega_n$, $D_1, \ldots, D_n \subset \mathbf{C}$ are bounded domains, and if $f\colon \Omega_1 \times \cdots \times \Omega_n \to D_1 \times \cdots \times D_n$ is a proper mapping, then there are a permutation σ of $\{1, \ldots, n\}$ and proper maps $f_j\colon \Omega_{\sigma(j)} \to D_j$ such that*

$$f(z_1, \ldots, z_n) = \big(f_1(z_{\sigma(1)}), \ldots, f_n(z_{\sigma(n)})\big).$$

EXAMPLE (THE BALL). The proper self-maps of the unit ball $\mathbf{B}^n = \{ z \in \mathbf{C}^n: |z_1|^2 + \cdots + |z_n|^2 < 1\}$, $n \geqslant 2$, were shown to be automorphisms by Alexander [3] (see also [71, 82]).

The existence of many proper mappings is given by a result of Grunsky [55] and Ahlfors [1].

THEOREM. *If M is a finite Riemann surface with nondegenerate boundary components, then there exists a proper mapping $f\colon M \to \Delta$.*

In general, however, given two Riemann surfaces M and N, it does not seem easy to say whether there exists a proper mapping $f\colon M \to N$.

Now we turn to the general structure of a proper mapping $f\colon \Omega \to D$, where $\dim \Omega = \dim D = n$. Two objects of interest are the Jacobian determinant

$$J_f(z) = \det\big(\partial f_i(z)/\partial z_j\big)$$

and the branch locus

$$V_f = \{ z \in \Omega: J_f(z) = 0\}.$$

Evidently, if $z_0 \in \Omega \setminus V_f$, then f is a local diffeomorphism in a neighborhood of z_0.

Some basic properties of a proper mapping $f\colon \Omega \to D$ are:

(A) *$f^{-1}(w_0)$ is a compact subvariety of Ω and is thus finite.*

(B) *f is an open mapping.*

(C) *f is not locally one-to-one in any neighborhood of any $z_0 \in V_f$.*

(D) *f has rank n on a dense open subset of Ω,*

(E) *the set of critical values $f(V_f)$ is a complex subvariety of D.*

For simple proofs of these facts, see Chapter 15.1 of Rudin [81].

It follows that $f\colon \Omega \setminus f^{-1}f(V_f) \to D \setminus f(V_f)$ is a proper, unbranched cover. If D is connected, then there is an integer m such that $f^{-1}(w)$ contains exactly m points for each $w \in D \setminus f(V_f)$. In fact, more is true:

(F) *$f^{-1}(w)$ contains m points if $w \notin f(V_f)$, and $f^{-1}(w)$ contains less than m points if $w \in f(V_f)$.*

Branching. We note two cases in which the branching behavior determines whether or not a mapping is biholomorphic. It is not hard to see that *if $f\colon \Omega \to D$ is an unbranched proper map, and if D is simply connected, then f is a biholomorphism.*

Less obvious is a result of Pinčuk [70]: *if $f\colon \Omega \to \Omega$ is an unbranched proper self-map, and if Ω has reasonable boundary, then f is a biholomorphism.*

Fornaess [48] has shown that there cannot be branching even at the boundary in the smooth, biholomorphic case: *if $f\colon \Omega \to D$ is a biholomorphic mapping between pseudoconvex domains with C^2 boundaries and if $f \in C^2(\overline{\Omega})$, then $J_f \neq 0$ on $\overline{\Omega}$ and $f^{-1} \in C^2(\overline{D})$.*

EXAMPLE (REINHARDT DOMAINS). Let Ω, D be Reinhardt domains, i.e. invariant under $(z_1,\ldots, z_n) \to (e^{i\theta_1}z_1,\ldots, e^{i\theta_n}z_n)$ for all $\theta_1,\ldots, \theta_n \in \mathbf{R}$. *If Ω_1, $\Omega_2 \subset\subset \mathbf{C}^n$ are Reinhardt domains satisfying*

(a) *$z_1 \cdots z_n \neq 0$ for $(z_1,\ldots, z_n) \in \overline{\Omega}_j$,*

(b) *$(z_1^{-1},\ldots, z_n^{-1}) \in \Omega_j$ if $(z_1,\ldots, z_n) \in \Omega_j$*

for $j = 1, 2$, *then every proper, unbranched cover* $f\colon \Omega_1 \to \Omega_2$ *is of the form* $f = (f_1, \dots, f_n)$ *with*

$$(*) \qquad f_j = c_j z_1^{\mu_j^1} \cdots z_n^{\mu_j^n} \quad \text{and} \quad \mu_j^k \in \mathbf{Z}.$$

This is Theorem 1 of **[9]**.

Further, *if* $\Omega \subset\subset \mathbf{C}^n$ *satisfies* (a), *and if* $f\colon \Omega \to \Omega$ *is a proper self-map, then* $f \in \operatorname{Aut}(\Omega)$, *and, in particular,* f *has the form* $(*)$. This is seen because $f\colon \Omega \setminus f^{-1}f(V_f) \to \Omega \setminus f(V_f)$ is an unbranched, proper cover. Thus for every $\sigma \in \pi_1(\Omega \setminus f(V_f))$, there exists $\gamma \in \pi_1(\Omega \setminus f^{-1}f(V_f))$ such that $f_*\gamma$ is an integer multiple of σ. Let T be the $n \times n$ matrix with integer coefficients which represents $f_*\colon H_1(\Omega, \mathbf{Z}) \to H_1(\Omega, \mathbf{Z})$ with respect to some basis. By the behavior of f_* on the fundamental group, it follows that $\det(T) \neq 0$. Our conclusion now follows from Theorem 2 of **[9]**.

For a rather general class of bounded Reinhardt domains Ω, D, Barrett [7] has shown that a proper map $f\colon \Omega \to D$ extends holomorphically to a neighborhood of $\bar{\Omega}$. (See also **[64, 65]**.)

EXAMPLE (COMPLETE, CIRCLED DOMAINS). *If* Ω, D *are circled and complete* (*i.e. invariant under* $(z_1, \dots, z_n) \to (\lambda z_1, \dots, \lambda z_n)$, $\lambda \in \mathbf{C}$, $|\lambda| \leqslant 1$), *then every proper map* $f\colon \Omega \to D$ *with* $f^{-1}(0) = \{0\}$ *is polynomial* (Bell **[21]**).

2. The analytic projection operator. The Bergman metric and kernel, while useful tools for biholomorphic mappings, are not invariant under proper mappings. The analytic (Bergman) projection operator, however, is invariant. This will be used in §3, where we describe the proof that proper mappings extend smoothly to the boundary.

If $\omega = h(z)\, dz_1 \wedge \cdots \wedge dz_n$ and $\eta = k(z)\, dz_1 \wedge \cdots \wedge dz_n$ are $(n, 0)$-forms on Ω, then we may consider the inner product

$$(\omega, \eta)_\Omega = i^{n^2} \int_\Omega \omega \wedge \bar{\eta},$$

which is well defined (as integration of a $2n$-form over a $2n$-manifold) independently of any metric on Ω. It is easily seen that

$$\|\omega\|_\Omega^2 = (\omega, \omega)_\Omega = 2^n \int_\Omega |h|^2\, dV,$$

where dV denotes the Euclidean volume. Let $L^2(\Omega)_{n,0} = \{(n, 0)\text{-forms } \omega \text{ with } \|\omega\|^2 < \infty\}$, and let $L_a^2(\Omega)_{n,0}$ denote the holomorphic n-forms in $L^2(\Omega)_{n,0}$. Thus $L_a^2(\Omega)_{n,0}$ is a closed subspace of $L^2(\Omega)_{n,0}$, and we may define the orthogonal projection

$$P_\Omega\colon L^2(\Omega)_{n,0} \to L_a^2(\Omega)_{n,0}.$$

We recall that if $f\colon \Omega \to D$ is a holomorphic mapping, then the pull-back operator f^* is defined by

$$f^*(h(w)\, dw_1 \wedge \cdots \wedge dw_n) = h(f(z)) J_f(z)\, dz_1 \wedge \cdots \wedge dz_n.$$

Thus, if $f\colon \Omega \to D$ is a p-to-1 proper mapping, then by the usual "change of variables" formula, we have

$$\int_\Omega |h(f(z))|^2 |J_f(z)|^2\, DV_z = p \int_D |h(w)|^2\, DV_w,$$

since $|J_f|^2$ is the same as the real Jacobian of f. It follows, then, that for $\eta \in L^2(\Omega)$,

$$\|f^*\eta\|_\Omega^2 = p\|\eta\|_D^2, \tag{2.1}$$

and thus $f^*\colon L^2(D)_{n,0} \to L^2(\Omega)_{n,0}$.

The relation between the holomorphic projection P and proper mappings was given by Bell [19].

LEMMA. *If $f\colon \Omega \to D$ is proper, then $P_\Omega f^* = f^* P_D$.*

PROOF. There are two cases to consider.

Case (i). If $\eta \in L_a^2(D)$, then $P_D\eta = \eta$. On the other hand, by (2.1), $f^*\eta \in L_a^2(\Omega)$, so $P_\Omega f^*\eta = f^*\eta$.

Case (ii). If $\eta \perp L_a^2(D)_{n,0}$, we must show $f^*\eta \perp L_a^2(\Omega)_{n,0}$. Let us first assume that $\eta = (\partial/\partial w_j)\phi(w)\, dw_1 \wedge \cdots \wedge dw_n$, where $\phi \in C_0^\infty(D \setminus f(V_f))$. If $g \in L_a^2(\Omega)_{n,0}$ then, setting $V = f^{-1}f(V_f)$, and denoting the local inverses of f by $F_1, \ldots, F_q$ we have

$$\begin{aligned}
&\int_{\Omega\setminus V} f^*\eta \wedge \bar{\omega}\\
&\quad = \int_{\Omega\setminus V} \frac{\partial}{\partial w_j}\phi(f(z))J_f(z) \wedge \overline{g(z)}\, Dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n\\
&\quad = \sum_{k=1}^{q} \int_{D\setminus f(V_f)} \frac{\partial}{\partial w_j}\phi(w)\, \overline{g(F_k(w))}\; \overline{J_{F_k}(w)}\, dw_1 \wedge \cdots \wedge dw_n\\
&\qquad\qquad \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n\\
&\quad = \sum_{k=1}^{q} \int_{D\setminus f(V_f)} \phi(w) \frac{\partial}{\partial w_j}\overline{\left(g(F_k(w))J_{F_k}(w)\right)}\, dw \wedge d\bar{w} = 0.
\end{aligned}$$

We conclude that $f^*\eta \perp L_a^2(\Omega)_{n,0}$.

Finally, the $(n,0)$-forms

$$\eta = \sum_{j=1}^{n} \frac{\partial\phi_j}{\partial w_j} dw_1 \wedge \cdots \wedge dw_n,$$

$\phi_j \in C_0^\infty(D \setminus f(V_f))$, are dense in $L_a^2(D)_{n,0}^\perp$. For if $h \in L^2(D)_{n,0}^\perp$ and $(h,\eta)_D = 0$, integration by parts shows that h is holomorphic on $D \setminus V$. By the Riemann Removable Singularity Theorem, $h \in L_a^2(D)_{n,0}$, and thus $h = 0$.

REMARK. A similar argument was given by Pinčuk [74] to show that the kernel functions themselves satisfy

$$K_\Omega(z) \geqslant (1/p)|J_f(z)|^2 K_D(f(z)).$$

Condition R. In the context of boundary regularity it is important to know whether the holomorphic projection preserves smooth functions, i.e.

$$P_\Omega\left(C_{n,0}^\infty(\bar{\Omega})\right) \subset C_{n,0}^\infty(\bar{\Omega}). \tag{2.2}$$

Domains with regular projection, i.e. which satisfy (2.2), are said to satisfy *Condition R.*

One approach to Condition R is given through the $\bar{\partial}$-Neumann problem. The $\bar{\partial}$-solving operator S corresponding to the $\bar{\partial}$-Neumann problem is given by the map (if it exists)

$$S\colon \{\, \mu \in L^2(\Omega)_{0,1} \colon \bar{\partial}\mu = 0\} \to L^2(\Omega)$$

such that $S(\mu)$ satisfies

$$(*) \qquad \bar{\partial} S(\mu) = \mu$$

and $S(\mu)$ minimizes $\int_\Omega |S(\mu)|^2\, dV$ over all solutions of $(*)$. If we identify $(n,0)$-forms with functions in the obvious way, then it is evident that

$$(**) \qquad f - P_\Omega f = S(\bar{\partial} f).$$

Thus Ω will satisfy Condition R if

$$(2.3) \qquad S\colon \{\, \mu \in C^\infty(\bar{\Omega})_{0,1} \colon \bar{\partial}\mu = 0\} \to C^\infty(\bar{\Omega}).$$

So far, the greatest source of domains known to satisfy Condition R is from condition (2.3), which is obtained from subelliptic estimates for the $\bar{\partial}$-Neumann operator. The Neumann operator N is an operator which inverts the Laplacian $\square = \sum \partial^2/\partial z_j \partial \bar{z}_j$ subject to the "$\bar{\partial}$-Neumann" boundary conditions. That is, for a $(0,1)$-form μ on Ω, $N\mu$ is a $(0,1)$-form solving $\square N\mu = \mu$ on Ω and satisfying the $\bar{\partial}$-Neumann conditions

$$N\mu \lrcorner \bar{\partial} r = 0, \qquad \bar{\partial}(N\mu) \lrcorner \bar{\partial} r = 0$$

on $\partial\Omega$. These boundary conditions are degenerate (noncoercive), but they serve as a substitute for the orthogonality condition corresponding to $(**)$. Finally, if the Neumann operator exists, there is a Hodge-type decomposition for N, from which we have $S\mu = \bar{\partial}^* N\mu$ for μ satisfying $\bar{\partial}\mu = 0$ (see the survey article of Kohn **[60]**).

The work of Kohn **[59]** showed that (2.3) holds under certain geometric hypotheses on $\partial\Omega$. Diederich and Fornaess **[36]** have proven, in particular, that these hypotheses are satisfied for pseudoconvex domains with real-analytic boundaries. D. Catlin **[32]** has shown, more generally, that (2.3) holds even for pseudoconvex domains which have finite type in the sense of D'Angelo **[95]**.

Although pseudoconvexity (or at least 1-convexity) is a necessary condition for (2.3), it is not clear what role pseudoconvexity plays in Condition R. Bell-Boas **[25]**, Bell **[23]** and Barrett **[5]** have shown that (2.2) holds in cases where (2.3) does not hold, i.e. where Ω is not pseudoconvex and $\bar{\partial}$ is not solvable. On the other hand, Barrett **[8]** has recently found a smoothly bounded but not pseudoconvex domain for which Condition R fails.

3. Boundary regularity. The following conjecture is well known but unresolved: *If Ω, $D \subset\subset \mathbf{C}^n$ are domains with smooth boundary, then every proper mapping $f\colon \Omega \to D$ extends smoothly to $\bar{\Omega}$.*

Before discussing some partial results in this direction, we give some examples of poorly behaved proper mappings.

Let $\Omega = \{(z,w) \in \mathbf{C}^2 \colon |z| < 1\}$, $\mathbf{T} = \mathbf{C}/(\mathbf{Z} + i\mathbf{Z})$, and $\Delta = \{\, z \in \mathbf{C} \colon |z| < 1\}$.

EXAMPLE. *If $f(z,w)\colon \Omega \to \Omega$ is given by $f(z,w) = (z, w + g(z))$, where $g(z)$ is any analytic function on Δ, then f is proper.*

EXAMPLE. *The function* $f\colon \Delta \times \mathbf{T} \to \Delta \times \mathbf{T}$ *given by* $f(z,w) = (z, w + g(z))$ *(as above) is proper.*

EXAMPLE. *The function* $G(z,w) = (z,(w - g(z))^2)$ *yields a proper mapping* $G\colon \Omega \to \Omega$ *and a proper mapping* $G\colon \Delta \times \mathbf{T} \to \Delta \times \mathbf{T}$.

In the first two examples, f does not extend smoothly to the boundary if g does not extend smoothly to $\bar{\Delta}$. In the third example, g may be chosen so that $V_G = \{ w = g(z) \colon z \in \Delta \}$ is dense in $\partial\Omega$ (or $\partial\Delta \times \mathbf{T}$).

EXAMPLE (PIECEWISE - SMOOTH BOUNDARY). Fridman [52] has shown that there is a domain $\Omega \subset\subset \mathbf{C}^2$ with piecewise-smooth boundary, and there is a biholomorphism $f\colon \Omega \to \Delta^2$ which does not extend continuously to $\bar{\Omega}$.

The first step to establishing boundary regularity is to show that boundary distances are preserved. For this we use a result of Diederich and Fornaess [35]: *if* $D \subset\subset \mathbf{C}^n$ *is a pseudoconvex domain with* C^2 *boundary, there exists* $r \in C^2(\bar{D})$ *with* $\{-r < 0\} = D$, $\{ r = 0\} = \partial D$, $\nabla r \neq 0$ *on* ∂D, *and* $-(-r)^\varepsilon$ *is plurisubharmonic on* D *for some* $\varepsilon > 0$. The following argument is due to Henkin and Pinčuk [57, 70, 76].

LEMMA 1. *If* Ω, $D \subset\subset \mathbf{C}^n$ *are pseudoconvex with* C^2 *boundary, and if* $f\colon \Omega \to D$ *is a proper mapping, then there exists* $\varepsilon > 0$ *such that*

$$\varepsilon(\operatorname{dist}(f(z), \partial D))^{1/\varepsilon} \leqslant \operatorname{dist}(z, \partial\Omega) \leqslant \varepsilon^{-1}(\operatorname{dist}(f(z), \partial D))^{\varepsilon}.$$

PROOF. Let r be the smooth defining function above. Then $|r(w)|$ is estimated above and below by $C\operatorname{dist}(w, \partial D)$. Since f is proper, $-(-r(f(z)))^\varepsilon$ is a psh. exhaustion function for Ω. By the Hopf Lemma, there is a constant $C > 0$ such that

$$-(-r(f(z)))^{\varepsilon} \leqslant -C\operatorname{dist}(z, \partial\Omega).$$

It follows, then, that

$$\operatorname{dist}(f(z), \partial D)^{\varepsilon} \geqslant C\operatorname{dist}(z, \partial\Omega).$$

The other inequality is obtained in the same manner because if ρ is a psh. exhaustion for Ω, then

$$\psi(w) = \max_{z \in f^{-1}(w)} \rho(z)$$

is a psh. exhaustion for D.

Now since $f(z) = (f_1(z), \dots, f_n(z))$ is bounded, the Cauchy estimates give

$$|D^\alpha f_j(z)| \leqslant M\operatorname{dist}(z, \partial\Omega)^{-|\alpha|}.$$

Thus from Lemma 1 and the Cauchy estimates, we obtain

LEMMA 2. $f^*(C_0^\infty(\bar{D})_{n,0}) \subset C_0^\infty(\bar{\Omega})_{n,0}$.

We will also need the following.

LEMMA 3. $P(C_0^\infty(\bar{D})_{n,0}) = P(C^\infty(\bar{D})_{n,0})$.

PROOF. For $\phi \in C^\infty(\bar{D})$, we want to find $\psi \in C_0^\infty(\bar{D})$ such that $P\phi = P\psi$. We let $r \in C^\infty(\bar{D})$ be a defining function for D. By a partition of unity, we may assume that the support of ϕ is small enough that $\partial r/\partial z_1 \neq 0$ on $\partial\Omega \cap \operatorname{supp} \phi$.

We construct ψ of the form

$$\psi = \phi - \frac{\partial}{\partial z_1}\left(\sum_{j=1}^{\infty} \chi_j r^j\right).$$

We may arrange to have $\psi \in C_0^\infty(\bar{D})$ since we may solve successively for χ_j. This starts as

$$\chi_1 \frac{\partial r}{\partial z_1} = \phi \quad (\operatorname{mod} O(r))$$

and proceeds inductively as

$$\phi - \frac{\partial}{\partial z_1}\left(\sum_{j=1}^{k-1} \chi_j r^j\right) = k r^{k-1} \chi_k \quad (\operatorname{mod} O(r^k)).$$

This determines the value of $\chi_j|_{\partial D}$ in terms of $\chi_1, \ldots, \chi_{j-1}$ and their derivatives on ∂D. Clearly $\chi_j \in C^\infty(\bar{D})$ may be chosen so that the summation converges in $C^\infty(\bar{D})$.

Finally, it is evident that $P_D\phi = P_D\psi$ since $\partial h/\partial z_1 \perp L_a^2(D)$ for all $h \in C^1(\bar{D})$ with $h = 0$ on ∂D.

THEOREM. *Let Ω, $D \subset\subset \mathbf{C}^n$ be smoothly bounded and pseudoconvex, and let Ω satisfy Condition* R. *Then if $f: \Omega \to D$ is proper, $J_f f_1^{\alpha_1} \cdots f_n^{\alpha_n} \in C^\infty(\bar{\Omega})$ for all integers $\alpha_1, \ldots, \alpha_n \geqslant 0$.*

PROOF. We set $\eta = w_1^{\alpha_1} \cdots w_n^{\alpha_n}\, dw_1 \wedge \cdots \wedge dw_n$. By the preceding lemma, there is a smooth form $\xi \in C_0^\infty(\bar{D})_{n,0}$ with $P_D\xi = \eta$. Further,

$$f^*\eta = f_1^{\alpha_1} \cdots f_n^{\alpha_n} J_f\, dz_1 \wedge \cdots \wedge dz_n.$$

But

$$f^*\eta = f^*P_D\xi = P_\Omega f^*\xi \in C^\infty(\bar{\Omega})_{n,0}$$

since $f^*\xi$ is a smooth form on $\bar{\Omega}$, and Ω satisfies Condition R. This completes the proof.

This theorem shows that $f(z)$ is smooth on a dense subset of the boundary, i.e. on $\{z \in \partial\Omega: J_f(z) \neq 0\}$. This is already a useful result: for instance, it follows from Webster [91] that *if $\partial\Omega$ and ∂D are, in addition, defined by polynomial equations, then f is algebraic.*

By the theorem above, the problem of showing that $f \in C^\infty(\bar{\Omega})$ is thus reduced to a problem of division, which was solved by Bell and Catlin [26] and Diederich and Fornaess [39]. They prove the finite vanishing of J_f at $\partial\Omega$ and then a Division Theorem, both of which are of independent interest and are stated here.

LEMMA (FINITE VANISHING OF THE JACOBIAN). *If $f: \Omega \to D$ is proper, and if Ω, $D \subset\subset \mathbf{C}^n$ are smoothly bounded and pseudoconvex, then J_f vanishes to only finite order at $\partial\Omega$, i.e. there is no $z_0 \in \partial\Omega$ such that $|J_f(z)| = O(|z - z_0|^m)$ for $m = 1, 2, 3, \ldots$.*

DIVISION THEOREM. *Let $\Omega \subset\subset \mathbf{C}^n$ be a domain with smooth boundary. If $u(z)$ and $f(z)$ are holomorphic on Ω, if $u \in C^\infty(\overline{\Omega})$ vanishes to finite order at $\partial\Omega$, and if $uf^j \in C^\infty(\overline{\Omega})$ for $j = 1, 2, 3, \ldots,$ then $f \in C^\infty(\overline{\Omega})$.*

From these results it follows that the conjecture stated at the beginning of this section is valid for pseudoconvex Ω satisfying Condition R. In a subsequent note [**16**], it was shown that these arguments apply also to domains Ω, D inside a Stein manifold. It has also been shown that the pseudoconvexity hypothesis may be dropped if both Ω and D satisfy Condition R (see Bell [**18**]).

These methods also yield extensions to a neighborhood of $\overline{\Omega}$ in some cases (see [**20, 6**]).

4. Generic branching. If $f\colon \Omega \to D$ is proper, then so is

$$f_0 = f|_V\colon V \to W, \tag{4.1}$$

where $V = V_f$ and $W = f(V_f)$. Certain kinds of local singular behavior are not possible for proper mappings. For instance, a proper mapping cannot look like $h(z,w) = (z, zw)$ near the origin, since $h^{-1}(0,0) = \{0\} \times \mathbf{C}$ has positive dimension. It does not seem to be known, however, just what varieties V, W and proper mappings $f_0\colon V \to W$ can arise as in (4.1) (cf. also the conjecture in [**87**]).

Although the exact branching behavior is not known (except in the case $\Omega = \mathbf{B}^n$, in §5), the generic branching behavior is easy to describe, and there is a relation between the generic branching order of a mapping at the boundary and the order of vanishing of the Levi form. The following is an elementary illustration of this.

EXAMPLE. If $f(z,w) = (z, w^2)$, then $f\colon \Omega_1 \to \Omega_2$ and $f\colon \Omega_2 \to \Omega_3$ are proper, where

$$\Omega_1 = \{|z|^2 + |w|^4 < 1\},$$
$$\Omega_2 = \{|z|^2 + |w|^2 < 1\},$$
$$\Omega_3 = \{|z|^2 + |w| < 1\}.$$

The branch locus in both cases is $V_f = \Delta \times \{0\}$. Note that Ω_1 is Levi flat at $\partial\Omega_1 \cap \overline{V}_f$, Ω_2 is strongly pseudoconvex at $\partial\Omega_2 \cap \overline{V}_f$, and Ω_3 is not smooth at $\partial\Omega_3 \cap \overline{V}_f$.

The point is that the branching order decreases the degree of Levi flatness. If the boundary is nowhere Levi flat (i.e. strongly pseudoconvex), then a branched mapping has a nonsmooth image.

We note that the nature of the nonsmooth image can never be as "nice" as piecewise-smooth according to the following result of Pinčuk [**74**].

THEOREM. *Let D, $G \subset \mathbf{C}^n$ $(n > 1)$ be bounded, pseudoconvex domains, G having boundary of class C^2, and D having piecewise C^2-smooth, but not smooth, boundary. Then there does not exist a proper mapping $g\colon G \to D$.*

Generic branch points. Given a point $z_0 \in V_f$, we may move it slightly so that z_0 is a regular point of V_f, the rank of f_0 is $n-1$ at z_0, and $f(V_f)$ is regular at

$f(z_0)$. Thus we may make holomorphic changes of coordinates in neighborhoods of z_0 and $f(z_0)$ so that $z_0 = f(z_0) = 0$, $V_f = \{z_n = 0\}$ near $z_0 = 0$, $f(V_f) = \{z_n = 0\}$ near $f(z_0) = 0$, and $f_0(z_1, \ldots, z_{n-1}) = (z_1, \ldots, z_{n-1})$ holds for $z \in V_f$ near $z_0 = 0$.

Since $\{f_n(z) = 0\} = \{z_n = 0\}$ holds in a neighborhood of z_0, we have

$$f_n(z) = \sum_{j=1}^{\infty} a_j(z') z_n^j,$$

where a_j is analytic in $z' = (z_1, \ldots, z_{n-1})$. Moving z_0 slightly within V_f, we may assume

$$f_n(z) = \sum_{j=M}^{\infty} a_j(z') z_n^j,$$

and $a_M(0) \neq 0$. Thus

$$f_n(z) = z_n^M U(z)$$

where $U(0) \neq 0$. By a change of coordinates in D, $z_n' = z_n(U(z))^{1/M}$, we may assume $f_n(z) = z_n^M$ in a neighborhood of z_0.

We conclude this discussion with the observation that *for a generic point $z_0 \in V_f$, holomorphic changes of coordinates may be made at z_0 and $f(z_0)$ so that*

$$f(z) = \left(z_1 + z_n g_1(z), \ldots, z_{n-1} + z_n g_{n-1}(z), z_n^M\right). \tag{4.2}$$

The *branching order* of f at z_0 is $\operatorname{Ord}(f, z_0) = M - 1$, and is constant on the irreducible component of V_f containing z_0.

REMARK 1. By a change of coordinates of the form

$$z_j' = z_j - g_j(0) z_n - \frac{\partial g_j}{\partial z_n}(0) z_n^2, \qquad 1 \leqslant j \leqslant n-1, \ z_n' = z_n,$$

we may assume (4.2) satisfies $g_j(0) = \partial g_j(0)/\partial z_n = 0$.

REMARK 2. If $\partial\Omega$ is strongly pseudoconvex and $f\colon \Omega \to D$, $f \in C^\infty(\overline{\Omega})$, is proper, then we can choose a generic $z_0 \in \overline{V}_f \cap \partial\Omega$ such that (4.2) holds to arbitrarily high order.

Next we give a measure of flatness of the boundary. If $r \in C^\infty(\overline{D})$, $D = \{r < 0\}$ and $\nabla r \neq 0$ on ∂D, then the determinant of the Levi form is given by

$$\lambda_D(w) = -\det\left(\begin{array}{c|c} 0 & \partial r/\partial \overline{w}_j \\ \hline \partial r/\partial w_i & \partial^2 r/\partial w_i \partial \overline{w}_j \end{array}\right) |\nabla r|^{-n-1}.$$

A point $w_0 \in \partial D$ is strongly pseudoconvex if the Levi form is positive definite and thus $\lambda_D(w_0) > 0$. If $f\colon \Omega \to D$ is proper, then $\rho = r \circ f$ is a defining function for Ω, and by the chain rule we may compute

$$\lambda_\Omega(z) = \lambda_D(f(z)) |J_f(z)|^2 \tag{4.3}$$

for $z \in \partial\Omega$. [The fact that ρ is a defining function, i.e. $\nabla\rho \neq 0$ on $\partial\Omega$, is seen, since by **[35]** we may take r to be a smooth defining function such that $-(-r)^{2/3}$ is psh. on D near $f(z)$ and then apply the Hopf Lemma to $-(-\rho)^{2/3}$.]

To illustrate the utility of (4.3) we give a result which was proved in various forms in [**29, 73, 20, 38**] for conclusion (i) and [**74, 18**] for conclusion (ii).

THEOREM. *If Ω, $D \subset\subset \mathbf{C}^n$ are smoothly bounded pseudoconvex domains and Ω is strongly pseudoconvex, then for any proper mapping $f\colon \Omega \to D$ it follows that*
(i) *f is a local biholomorphism;*
(ii) *D is strongly pseudoconvex.*

PROOF. By §1, if f is not a local biholomorphism at some point of Ω, then $V_f \neq \varnothing$, and thus $\overline{V}_f \cap \partial\Omega \neq \varnothing$. By §3, $J_f \in C^\infty(\overline{\Omega})$ and thus $J_f(z_0) = 0$ for $z_0 \in \overline{V}_f \cap \partial\Omega$. By (4.3) it follows that $\lambda_\Omega(z_0) = 0$ and thus Ω is Levi flat at z_0, which is a contradiction.

It also follows from (4.3) that $\lambda_D \neq 0$ at $f(z_0)$, and thus D is strongly pseudoconvex at all points $f(\partial\Omega) \subset \partial D$. Since $f \in C(\overline{\Omega})$ and f is proper, $f(\partial\Omega) = \partial D$, which proves (ii).

From (4.3) it is evident that λ is not a biholomorphic invariant of the boundary, although the condition $\lambda = 0$ is invariant. If $z_0 \in \partial\Omega$, then we may define

$$\begin{aligned}\tau(z_0) &= \text{order of vanishing of } \lambda(z) \text{ at } z = z_0\\ &= \min\{\, m\colon \text{for every tangential differential operator}\\ &\qquad\qquad P \text{ on } \partial\Omega \text{ of order } m,\ P(\lambda(z_0)) = 0\},\end{aligned}$$

and τ is a biholomorphic invariant. A more thorough discussion of the invariants of the Levi form is given in [**95**]. From (4.2) and (4.3) we obtain a relation between the generic branching order and τ.

PROPOSITION. *If $f\colon \Omega \to D$ is a proper mapping, $f \in C^\infty(\overline{\Omega})$, and $z_0 \in \partial\Omega$ is a generic branch point, then $\tau(z_0) + 2 = (\mathrm{Ord}(f, z_0) + 1)(\tau(f(z_0)) + 2)$.*

If $\partial\Omega$ is real-analytic, we may make a semianalytic stratification of the set $\{ z \in \partial\Omega\colon \lambda(z) = 0\}$ into sets where τ is constant. From this we may extract a finite number of connected, real-analytic manifolds $\Gamma^1, \ldots, \Gamma^k$ with the properties:

(1) $\tau = \tau_j$ is constant on Γ^j, $1 \leqslant j \leqslant k$;
(2) there is a connected complex $(n-1)$-manifold $\tilde{\Gamma}^j \subset \mathbf{C}^n$ with $\Gamma^j \subset \tilde{\Gamma}^j$, $1 \leqslant j \leqslant k$.

The details of this construction are given in [**11**]. The theorem below shows that if Ω has real-analytic, pseudoconvex boundary, then there are only finitely many possibilities for V_f, which are determined by $\partial\Omega$. Further, for a proper mapping $f\colon \Omega \to D$ between two domains with real-analytic boundaries, the special manifolds in $\partial\Omega$ are taken onto the special manifolds of ∂D.

THEOREM. *Let $\Omega \subset\subset \mathbf{C}^n$ be a pseudoconvex domain with smooth, real-analytic boundary. Let $D \subset\subset \mathbf{C}^n$ be a domain with smooth boundary, and let $f\colon \Omega \to D$ be a proper mapping. Then for each irreducible component V of V_f there exists a point $z_0 \in \Gamma^j$ and an open set U containing z_0 such that $V \cap U = \tilde{\Gamma}^j \cap U$. Further, if ∂D is real analytic, and Γ' is an element of the stratification of ∂D, then there exist $\Gamma^{i_1}, \ldots, \Gamma^{i_r} \subset \partial\Omega$ such that $\Gamma^{i_1} \cup \cdots \cup \Gamma^{i_r}$ is dense in $f^{-1}(\Gamma')$.*

Two applications of this theorem are given below. Since the corresponding values of τ must be related as in the above proposition, we have the following application to a specific case, which was also treated by Landucci **[61]**.

THEOREM. *If* $\Omega = \{|z|^2 + |w|^{2p} < 1\}$ *and* $D = \{|z|^2 + |w|^{2q} < 1\}$, *then there exists a proper mapping* $f\colon \Omega \to D$ *if and only if* q *divides* p.

If the theorem is applied to the iteration of a self-mapping, we obtain the following, which was proved in **[11, 12, 24]**.

THEOREM. *If* $\Omega \subset\subset \mathbf{C}^n$ *is a pseudoconvex domain with smooth, real-analytic boundary, then every proper self-mapping* $f\colon \Omega \to \Omega$ *is an automorphism.*

5. Factorization. Let $f\colon \Omega \to D$ be a proper mapping. We will discuss the existence of a subgroup $\Gamma_f \subset \mathrm{Aut}(\Omega)$ of the group of automorphisms with the properties:

(i) $fg(z) = f(z)$ for all $z \in \Omega$,

(ii) $f^{-1}f(z) = \bigcup_{g\in\Gamma_f} g(z)$ for all $z \in \Omega$.

The existence of the group Γ_f gives a canonical factorization $f = \tilde{f}\eta$, where $\eta\colon \Omega \to \Omega/\Gamma_f$ is the quotient mapping, and $\tilde{f}\colon \Omega/\Gamma_f \to D$ is a biholomorphism.

A factorization does not always exist. For example, let $\Omega = D = \Delta$ be the unit disk in $\mathbf{C}$, and let $f\colon \Delta \to \Delta$ be given by

$$f(z) = \left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)^2\left(\frac{z-\beta}{1-\bar{\beta}z}\right),$$

where $\alpha, \beta \in \Delta$, $\alpha \neq \beta$. Note that $f^{-1}(0) = \{\alpha, \beta\}$, but that f has different branching orders at α and β. Thus there cannot exist $g \in \mathrm{Aut}(\Delta)$ satisfying (i) above and $g(\alpha) = \beta$.

The existence of a factorization of a proper map $f\colon \Omega \to D$ is easily seen to be equivalent to the unbranched covering $f\colon \Omega\setminus f^{-1}f(V_f) \to D\setminus f(V_f)$ being normal. The reason for this is that Γ_f are covering transformations by (i), and by (ii) the covering transformations are transitive.

If $f\colon \Omega \to D$ is generically p-to-1, and if f can be factored, then it is evident that the order of the group Γ_f is p. From §1,

$$V_f = \{ z \in \Omega\colon \text{the number of elements of } f^{-1}f(z) \text{ is } < p\}.$$

It follows that if f can be factored, then

$$V_f = \bigcup_{\substack{q\in\Gamma_f\\ g\neq \mathrm{id}}} \mathrm{Fix}(q),$$

where $\mathrm{Fix}(g) = \{ z \in \Omega\colon q(z) = z\}$ is the set of fixed points of g.

It is useful to know that in certain cases factorizations do exist.

THEOREM (FACTORIZATION). Let $\Omega \subset\subset \mathbf{C}^n$, $n \geqslant 2$, *be a simply connected, strongly pseudoconvex domain with* C^∞ *smooth boundary. If* $f\colon \Omega \to D$ *is proper, then* f *can be factored, i.e. there exists a subgroup* $\Gamma_f \subset \mathrm{Aut}(\Omega)$ *satisfying* (i) *and* (ii).

Proofs of this are given in [**13, 15**]. Since those proofs use holomorphic correspondences, which we have not discussed here, let us give a sketch of the proof in [**10**], which assumes $f \in C^\infty(\overline{\Omega})$. We take $z_0 \in \Omega \setminus f^{-1}f(V_f)$ and $z_1 \in f^{-1}f(z_0)$ and consider the germ of a covering transformation taking z_0 to z_1. It must be shown that this transformation, which is a branch of $f^{-1}f$, may be analytically continued over $f^{-1}f(V_f)$. The problem reduces itself to showing that if $p_1, p_2 \in f^{-1}f(V_f)$ are generic branch points, then $\mathrm{Ord}(f, p_1) = \mathrm{Ord}(f, p_2)$. To show this we let $\partial\Omega^j$ and f^j denote the germs of $\partial\Omega$ and f at p_j. We may assume $p_j \in \partial\Omega$ and f^j has the form (4.2), Remark 1. By a direct calculation, then, we see that if $(f^2)^{-1}f^1$ maps $\partial\Omega^1$ and $\partial\Omega^2$ then $M_1 = M_2$, i.e. the orders are the same.

This theorem has some implications for the construction of proper mappings. Let us recall one of the standard methods of constructing proper maps (see Rudin [**81**, p. 301]). *Let f: $\tilde{\Omega} \to \tilde{D}$ be a holomorphic mapping such that $f^{-1}(w)$ is compact for $w \in \tilde{D}$. Then for any $p \in \tilde{\Omega}$ there is an arbitrarily small connected neighborhood Ω of p such that f: $\Omega \to f(\Omega)$ is proper*. The domain Ω is obtained by taking a connected component of $f^{-1}(D)$ where D is a neighborhood of $f(p)$, which may be chosen with great freedom. In contrast, there is not much freedom in choosing Ω if f is not locally biholomorphic at p. If we wish to choose Ω to be strongly pseudoconvex, then (if $p \in V_f$) Ω must have a nontrivial automorphism group. On the other hand (see [**30, 53**]), a generic Ω has no automorphisms.

It is evident that the Factorization Theorem above may be used to reduce the study of the branch locus to the study of Γ_f. Thus far Γ_f has been identified only in the case $\Omega = \mathbf{B}^n$.

A finite subgroup $G \subset U(n)$ of the unitary group is called a *reflection group* if it is generated by its reflections, i.e. transformations which are conjugate to a diagonal matrix of the form

$$\begin{pmatrix} e^{2\pi i/k} & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

It is a classical theorem of Chevalley that the invariant polynomials of a reflection group are generated by n homogeneous polynomials $p_1, \ldots, p_n$, which we denote by $P_\Gamma = (p_1, \ldots, p_n)$. Examples of P_Γ are given in Rudin [**83**], where the following is proved.

THEOREM. *Let f: $\mathbf{B}^n \to D$, $n \geqslant 2$, be a proper mapping. Then Γ_f has the following properties*:

(i) *there exists $T \in \mathrm{Aut}(\mathbf{B}^n)$ such that $0 \in \mathbf{B}^n$ is a fixed point of $T^{-1}\Gamma_f T$.*

(ii) *$T^{-1}\Gamma_f T \subset U(n)$ is a finite reflection group.*

(iii) *the mapping f may be factored as $f = \tilde{f}P_\Gamma T$, where $\tilde{f}$: $P_\Gamma(\mathbf{B}^n) \to D$ is a biholomorphism.*

Since the fixed point set of a reflection is a complex linear hyperplane, it follows that $J_F = TH_1 \cup \cdots \cup TH_k$, where H_j is a complex hyperplane passing through the origin, and T is from the theorem above.

Let us remark that in Rudin's Theorem, the hypothesis that D is a nonsingular complex manifold is crucial. For if $G \subset \mathrm{Aut}(\mathbf{B}^n)$ is any finite subgroup, then $\mathbf{B}^n/G$ has the structure of a complex space (possibly with singularities), see [31]. Thus for the quotient $f: \mathbf{B}^n \to \mathbf{B}^n/G$ we have $\Gamma_f = G$, if the image $D = \mathbf{B}^n/G$ is allowed to be singular.

6. Mappings into higher-dimensional spaces. Here we present some results concerning proper maps $f: \Omega \to D$ with $\dim \Omega < \dim D$, but so little is known that we also mention some open problems. Although most of these problems have been known for some time, it seems worthwhile to state them all in the same place.

A. *Convex imbedding theorem.* The following result of J. E. Fornaess [47] and Henkin [96] is useful, e.g. in approximation results.

THEOREM. *If $\Omega \subset\subset \mathbf{C}^n$ is strongly pseudoconvex, then there exists a proper imbedding $f: \Omega \to D \subset\subset \mathbf{C}^n$, where D is a strongly convex domain, and $f(\Omega)$ intersects ∂D transversely.*

A natural question that arises is: Can D be replaced by the ball $\mathbf{B}^N$? A result of J. Faran [46] indicates that this is unlikely. Faran's result is: *there is a real-analytic strongly pseudoconvex hypersurface $\Gamma \subset \mathbf{C}^n$ and a point $p \in \Gamma$ such that there is no holomorphic imbedding of a neighborhood of p in Γ into $\partial \mathbf{B}^N$ for any $N < \infty$.*

On the other hand, L. Lempert [63] has shown that Ω may be properly imbedded in the unit ball in Hilbert space.

B. *Imbedding in $\mathbf{C}^N$.* It is a basic result of Bishop, Narasimhan, and Remmert [28, 68, 77] that every n-dimensional Stein manifold M can be properly imbedded in $\mathbf{C}^N$ for $N = 2n + 1$. The problem is to make N as small as possible. By Bishop [28] there is a holomorphic mapping $f: M \to \mathbf{C}^n$ which is "almost proper" (i.e. all connected components of $f^{-1}(K)$ are compact if K is compact). Also by [28] there is a proper mapping $f: M \to \mathbf{C}^{n+1}$.

Forster [49] has shown that M may be properly imbedded in $\mathbf{C}^N$ with $N = [5n/3] + 2$. It was announced in [54] that M may be immersed in $\mathbf{C}^N$, properly immersed in $\mathbf{C}^{N+1}$, and properly imbedded in $\mathbf{C}^{N+2}$, with $N = [3n/2]$; but the details of the proofs have not yet appeared. For topological reasons, M cannot be imbedded in $\mathbf{C}^N$, $N = [3n/2]$. It is an open question whether an imbedding is possible with $N = [3n/2] + 1$.

A specific case of this question is whether it is possible to imbed an open Riemann surface properly in $\mathbf{C}^2$. This imbedding is possible in the cases of the disk, punctured disk, and annulus (see [4, 62]).

C. *Boundary regularity of imbeddings.* There are proper imbeddings $f: \Delta \to \mathbf{B}^2$ with $f \notin C(\bar{\Delta})$. To give an example, we choose $\phi_1, \phi_2 \in C(\partial\Delta)$ such that $e^{2\phi_1} + e^{2\phi_2} = 1$. We may choose ϕ_1, ϕ_2 such that the harmonic conjugates ϕ_1^*, ϕ_2^* are not continuous, but in any case $f = (\exp(\phi_1 + i\phi_1^*), \exp(\phi_2 + i\phi_2^*))$ is proper.

The problem remains, however, as to whether a proper imbedding (or merely proper mapping) $f: \Omega \to D$ extends smoothly to $\overline{\Omega}$ if Ω, D are both smooth and strongly pseudoconvex, and $\dim \Omega \geqslant 2$.

Results in this direction, involving generalizations of the Reflection Principle, have been obtained by Webster [93] and Cima, Krantz and Suffridge [33].

D. *Mapping balls to balls*. The following was obtained by S. Webster [92].

THEOREM. *If* $f: \mathbf{B}^n \to \mathbf{B}^{n+1}$, $n \geqslant 3$, *is proper*, $f \in C^3(\overline{\mathbf{B}}^n)$, *and* f *gives an immersion of* $\partial\mathbf{B}^n$ *into* $\partial\mathbf{B}^{n+1}$, *then there exists* $\psi \in \operatorname{Aut}(\mathbf{B}^{n+1})$ *such that* $\psi f(z_1,\dots,z_n) = (z_1,\dots,z_n,0)$.

Since the case $\mathbf{B}^1 \to \mathbf{B}^2$ is impossible, it is perhaps not surprising that $\mathbf{B}^2 \to \mathbf{B}^3$, the "borderline" case, is more complicated.

THEOREM (FARAN [44]). *Let* $f: \mathbf{B}^2 \to \mathbf{B}^3$ *be a proper map with* $f \in C^3(\overline{\mathbf{B}}^2)$. *Then there are* $\psi_2 \in \operatorname{Aut}(\mathbf{B}^2)$ *and* $\psi_3 \in \operatorname{Aut}(\mathbf{B}^3)$ *such that* $\psi_3 f \psi_2$ *is one of the following*:

(1) $(z,w) \to (z^3, w^3, \sqrt{3}\,zw)$,
(2) $(z,w) \to (z, zw, w^2)$;
(3) $(z,w) \to (z^2, \sqrt{2}\,zw, w^2)$;
(4) $(z,w) \to (z,w,0)$.

The regularity hypothesis $f \in C^3(\overline{\mathbf{B}}^n)$ of Webster's Theorem has been weakened in [34, 33], but it is not known whether any regularity hypothesis is necessary.

If $f: \mathbf{B}^n \to \mathbf{B}^k$ is proper, $k \leqslant 2n - 2$, and if f is holomorphic in a neighborhood of $\overline{\mathbf{B}}^n$, then Faran has shown that f is linear fractional (see [34, 45]).

E. *Mapping the ball to the polydisk*. It is known that there is no proper mapping between polydisk Δ^n and ball $\mathbf{B}^n$, e.g., [80]. There are generalizations of this showing that a polyhedron cannot be properly mapped to a domain with strongly pseudoconvex points [57, 58, 51]. Although it is easily seen that Δ^n cannot be mapped properly to $\mathbf{B}^N$, for any $N > 0$, it is unknown whether it is possible to map $\mathbf{B}^n$ properly to Δ^N (of course $N > n$).

F. *Mappings which are nearly proper*. It is of interest to consider mappings $f: \Omega \to D$ which are close to being proper. For instance, the mapping of the disk $f: \Delta \to D$, which realizes the Kobyashi metric, is such a map (see E. Poleckii [75]).

Given a mapping $f: \Omega \to D$, we may consider the exceptional set

$$E = \left\{ z_0 \in \partial\Omega \colon \limsup_{z \to z_0} \operatorname{dist}\big(f(z), \partial D\big) > 0 \right\}.$$

If E is small, is f necessarily proper? This seems not to be known even in the simplest cases, e.g., when E is a point and $\dim \Omega = \dim D \geqslant 2$. (The case $\dim \Omega = 1$ is, in general, false.) It is also possible to formulate the same question in terms of the radial exceptional set

$$E_{\text{rad}} = \left\{ z_0 \in \partial\Omega \colon \limsup_{\varepsilon \to 0} \operatorname{dist}\big(f(z_0 - \varepsilon N(z_0)), \partial D\big) > 0 \right\},$$

where $N(z_0)$ denotes the outward normal to $\partial\Omega$ at z_0. A Baire category argument (cf. Sadullaev **[84]**) may be given to show that if $E_{\text{rad}} = \varnothing$, then $E \subset \partial\Omega$ is a nowhere dense F_σ-set.

To illustrate this question we consider mappings $f\colon \Omega \to \Delta$ with $\dim \Omega \geqslant 2$, which can in no sense be close to proper. The condition that E_{rad} has zero measure in $\partial\Omega$ is equivalent to f being an "inner" function. Since inner functions have been shown to exist **[2, 56, 67]**, the condition that $|E_{\text{rad}}| = 0$ is too weak. The other extreme is the case $E_{\text{rad}} = \varnothing$, in which case Sadullaev **[84]** has shown that such f do not exist.

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DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401