how modular representations could be used to prove structure theorems for p-solvable groups. (In fact, the first version of their theorem showed how to give the structure of finite groups of exponent six by a reduction to examination of specific representations of the symmetric group of degree three and the alternating group of degree four in characteristics two and three, respectively!)

The last volume is devoted to simple groups and begins with the local theory of finite groups. A p-local subgroup of a finite group is a normalizer of a nonidentity p-subgroup, and the local theory is that body of theorems which show how much of the structure of finite groups is captured by these subgroups. These ideas are the main ideas used in the classification. The next two chapters deal with certain permutation groups which are basic to the study of simple groups. First, the Zassenhaus groups are described, and this is the first time in book form. This is one of the first steps of the classification. Second, multiply transitive groups are studied; this is where the sporadic groups first arose, in the work of Mathieu, well over a century ago.

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Factorization of matrix functions and singular integral operators, by K. Clancey and I. Gohberg, Operator Theory: Advances and Applications, Vol. 3, Birkhäuser Verlag, Basel, 1981, x + 234 pp., $17.55. ISBN 3-7643-1297-1

The present book is a review of the current state of the theory of factorization of nonsingular matrix functions along closed contours and systems of singular integral equations. As such it is a natural companion of the Gohberg-Krupnik book [8] which deals with scalar equations. To understand the type of factorization considered here, it is necessary to recall its definition. Let Γ be an oriented closed smooth contour on the Riemann sphere with inner domain F^+ and outer domain F^-, and assume $t^{\pm}$ are fixed points in $F^{\pm}$. Let $A(t)$, $t \in \Gamma$, be a nonsingular $n \times n$ matrix whose entries are continuous functions on Γ. A *factorization of A relative to the contour* Γ is a representation of A in the form

$$(1) \qquad A(t) = A_-(t)D(t)A_+(t), \qquad t \in \Gamma,$$

where $D(t)$ is an $n \times n$ diagonal matrix,

$$(2) \qquad D(t) = \operatorname{diag}\left(\left(\frac{t-t^+}{t-t^-}\right)^{\kappa_1}, \ldots, \left(\frac{t-t^+}{t-t^-}\right)^{\kappa_n}\right),$$

the matrix functions A_+ and A_- are analytic on the inner and outer domain of Γ, respectively, both A_+ and A_- are continuous up to the boundary of Γ, and $\det A_{\pm}(t)$ does not vanish on $F^{\pm} \cup \Gamma$. The integers $\kappa_1 \geqslant \cdots \geqslant \kappa_n$ are uniquely determined by A and called the *partial indices* of A relative to Γ. The factorization is called *canonical* when all indices are zero. The role of the points

t^+, t^- is not important as long as they are in the right domains. Often the contour Γ will be just the unit circle Γ_0, and in that case one may take $t^+ = 0$, $t^- = \infty$ and

$$D(t) = \operatorname{diag}(t^{\kappa_1}, \dots, t^{\kappa_n}).$$

The fact that the diagonal term $D(t)$ appears as the middle factor in the right-hand side of (1) is crucial. It accounts for the uniqueness of the partial indices, which is lost in factorizations of the form $A = A_- A_+ D$, where D appears as the third factor.

Factorizations of matrix functions relative to a contour were introduced in the beginning of this century by J. Plemelj, N. I. Muskhelishvili, N. P. Vekua, and others as special solutions of barrier problems of Hilbert and Riemann-Hilbert type in complex function theory. Later on other reasons appeared which justified interest in such factorizations: Systems of singular integral equations, vector-valued Wiener-Hopf equations on a half line and their discrete analogues—the block Toeplitz equations—can be solved when a factorization of their symbol (which is a matrix function) is available (see part VI of [**13**] and [7, **6**]). In this development the Gohberg-Kreĭn paper [7] played an important role. In [7] the classical factorization results were extended to wider and more natural classes of matrix functions and were put into the context of functional analysis and operator theory. This new approach made it possible to use a great variety of methods and results from Banach algebra theory and opened the way for new applications to integral equations and operator theory. Today it is clear that applications of factorization relative to a contour are not restricted to barrier problems and systems of integral equations. Such factorizations have become increasingly important, and now the subject seems to form an intersection where different routes in pure and applied mathematics meet. See, for example, Grothendieck's paper [**5**] where the factorization appears in disguised form and [**2**, **4**, **9**, **10**, **12**, **14**] for recent developments and further references.

In the present book the emphasis is on the connections between factorization relative to a contour and singular integral operators. The interplay between the two is one of the main themes. To see these connections in more detail consider on $L_p^n(\Gamma)$ (the space of all $\mathbf{C}^n$-valued vector functions with components in $L_p(\Gamma)$) the operator T defined by

$$(3) \qquad (Tf)(s) = B(s)f(s) + C(s)\frac{1}{\pi i}\int_\Gamma \frac{f(t)}{t-s}\,dt, \qquad s \in \Gamma.$$

Here B and C are given continuous $n \times n$ matrix functions defined on Γ. In other words, $T = B + CS$, where B and C are viewed as multiplication operators and S is the basic operator of singular integration on the contour Γ. Usually the operator T is rewritten in the form $T = (B - C)[AP + Q]$, where $A = (B - C)^{-1}(B + C)$ and P, Q are the projections $P = \frac{1}{2}[I + S]$, $Q = \frac{1}{2}[I - S]$. If, in addition, the matrix function A admits a factorization $A = A_- D A_+$ relative to Γ, then T can be written as the product

$$(4) \qquad T = (B - C)A_-[DP + Q][A_+P + A_-^{-1}Q].$$

In this representation of T the factors $(B - C)A_-$ and $A_+P + A_-^{-1}Q$ are invertible operators, and hence the problem of inversion of T is reduced to the equivalent problem of inversion of the simpler diagonal operator $DP + Q$. For example, the operator T is invertible only in the case of canonical factorization when all partial indices are equal to zero, and in this case

$$(5) \qquad T^{-1} = \left(A_+^{-1}PA_-^{-1} + A_-QA_-^{-1}\right)(B - C)^{-1}.$$

In general, when nonzero partial indices occur, the operator T is Fredholm, the dimension of its kernel is equal to the sum of the absolute values of the negative indices, and the codimension of the image of T equals the sum of the positive indices.

For singular integral operators with coefficients B and C that are not continuous, but merely bounded and measurable, similar connections hold, except now the factorization (1) with continuous factors has to be replaced by a so-called generalized factorization (H. Widom (1960), I. B. Simoneko (1968) and N. Y. Krupnik (1976)). In the generalized factorization the factors $A_\pm$ appearing in (1) are allowed to be unbounded; they only have to satisfy the requirement that the operator $A_-QA_-^{-1}$ acts as a bounded operator on $L_p^n(\Gamma)$, which is a natural condition if one looks at the form of the inverse of the singular integral operator T in (5). It is clear that generalized factorization depends on the contour Γ and on the space where the singular integral operator acts, and hence one has to speak about *generalized factorization relative to* $L_p(\Gamma)$. To distinguish generalized factorization from the factorization with continuous factors considered in (1) the latter will be referred to as *continuous factorization*. Continuous, as well as generalized, factorization is defined in global terms, but the existence of such factorization is basically determined by the *local properties* of the function.

An important special topic in factorization theory concerns *selfadjoint matrix functions* on the unit circle Γ_0. For such a function it is more natural to consider the factorization

$$(6) \qquad A(t) = A_+^*(t)D_0(t)A_+(t), \qquad t \in \Gamma_0,$$

where A_+ has the usual properties and $D_0(t)$ is a selfadjoint block matrix function of the following form: the blocks off the second diagonal are all zero and the blocks on the second diagonal are:

$$t^{-\alpha_1}I_{n_1}, \ldots, t^{-\alpha_r}I_{n_r}, \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, t^{\alpha_r}I_{n_r}, \ldots, t^{\alpha_1}I_{n_1}.$$

Here the symbol I_k denotes the $k \times k$ identity matrix. A relatively recent theorem by A. M. Nikolaichuk and I. M. Spitkovskiĭ (1975) states that a selfadjoint factorization as in (6) is always possible when A admits an ordinary factorization relative to Γ_0. Moreover in that case $\alpha_1, \ldots, \alpha_r$ are the distinct positive indices, the index α_j occurring n_j times, and p and q are positive integers such that $p + q = n - 2\Sigma_{j=1}^r n_j$ and $p - q$ is the signature of the matrix $A(t)$ which does not depend on $t \in \Gamma$ (a fact which is obvious for the continuous case and requires a little proof in the general case).

A full presentation of all these topics—continuous and generalized matrix factorization, the connections with and the applications to singular integral operators on vector-valued L_p-spaces, selfadjoint matrix factorizations, local principles—which over the past 25 years required the work of many mathematicians, appears here in book form for the first time. But there is more. In connection with the method of inversion of the singular integral operator, which has been described above, many problems arise, which to a certain extent involve the measure in which the method is effective. The book pays much attention to these problems and the interesting developments which originated in this way. Here three of them will be discussed in more detail.

A first problem is that for continuous matrix functions a continuous factorization may not exist at all. The main obstacle is that, in general, the algebra of continuous functions on the curve Γ is not decomposing, i.e., it is not always possible to write a continuous function on Γ as the sum of two continuous functions, one of which has an analytic extension inside Γ, while the other has an analytic extension outside Γ. The connection between continuous factorization and decomposing algebras of matrix functions, which has its origins in the work of the second author, is nicely described in Chapter 2 of the present book with several new points added. In general, if the continuous matrix function A is nonsingular and sufficiently smooth, then A admits a continuous factorization relative to Γ. For generalized factorization the situation is different. A nonsingular continuous matrix function always admits a generalized factorization relative to $L_p(\Gamma)$ $(1 < p < \infty)$ and, which is less obvious, the factorization can be made independent of p in the range $1 < p < \infty$. In particular, by the Nikolaichuk-Spitkovskiĭ theorem mentioned above, a continuous selfadjoint matrix function on the unit circle admits a generalized selfadjoint factorization of the form (6) (cf. **[1]** where recently this result has been reproved in the context of Beurling-Lax theorems for shifts on an indefinite metric space). If the entries of the matrix function A are not continuous, but only piecewise continuous, then, in general, generalized factorization relative to $L_p(\Gamma)$ is possible only for certain values of p and the partial indices depend on p. The latter results are due to I. Gohberg and N. Ya. Krupnik (1969) and build on earlier work for scalar functions; they appear in Chapter 8 of the present book.

A second main problem to which the book pays much attention and which arises naturally in applications is the problem of finding the factors and the indices in an explicit way. This problem appears in several different versions. A preliminary version concerns canonical factorization. Assuming the matrix function admits factorization, continuous or generalized, when can one tell in advance that all partial indices will be zero and the corresponding singular integral operator will be invertible? A classical case is that of dissipative matrix functions on the unit circle. When on the unit circle such a factorization exists, it is necessarily canonical. On the other hand, a (somewhat unexpected) theorem of A. S. Marcus and V. I. Macaev (1976) (appearing here for the first time in English) tells us that in the matrix case this result characterizes the unit circle and does not hold for other contours. In this context also, Rabindranathan's theorem (1969) should be mentioned (appearing in Chapter

8), which gives a necessary and sufficient condition in order that an essentially bounded nonsingular matrix function on the unit circle Γ_0, whose inverse is also essentially bounded, admits a canonical factorization relative to $L_2(\Gamma_0)$.

Finding the factors and partial indices explicitly is in general a very complicated and nontrivial problem which has solutions in special cases only. One of these cases concerns rational matrix functions. The classical way to obtain the factorization of a rational matrix function with no poles and zeros on the contour Γ (which is reviewed in Chapter 1) is based on an algorithm which produces the factors in a finite number of steps, but this algorithm does not yield formulas. In **[2]**, which is briefly mentioned in the present book, the concept of realization of a rational matrix function is used to get necessary and sufficient conditions for canonical factorization expressed in geometric terms. The work **[2]** also provides explicit formulas for the factors in a canonical factorization, which recently have been extended to the noncanonical case **[3]**.

Triangular matrix functions form another class for which it is possible to obtain explicit factorizations. They form the main topic of Chapter 4. Here one finds the constructive procedure (due to Gohberg and Kreĭn **[7]**) to obtain a continuous factorization of a nonsingular triangular matrix function with continuous entries from a decomposing algebra and the general rule of G. N. Chebotarev (1956) to get the partial indices of a 2×2 triangular matrix function. Also in Chapter 4 one finds the Gohberg-Lerer results (1978) about mixed triangular 2×2 matrix functions on a compound contour. Such a function is lower triangular on one component of the contour and triangular with respect to the second diagonal on the other component. The interesting point is that its indices can be expressed in terms of the classical resultant for scalar polynomials. Further work in this direction appears in the dissertation of B. Kon **[11]**.

A third main problem is the behaviour of the indices and the factors when the matrix function A is perturbed. This is the main topic of the last chapter. In the context of generalized factorization a proof is given of the Gohberg-Kreĭn theorem **[7]** which states that the partial indices are stable under small perturbations only when the difference between the largest and the smallest is at most one. (In systems theory the latter condition reappears in the description of structural stability of systems **[15]**.) Also, matrix functions $A(\cdot, s)$, which depend on a second parameter s, are investigated. First for the case when the dependence on s is analytic, and, secondly, for the case when $A(t, s)$ is rational in (t, s). Assuming that the latter holds true, let $\Phi(A)$ be the set of all s such that $\det A(t, s) \neq 0$ for each $t \in \Gamma$. For $s \in \Phi(A)$ the function $A(\cdot, s)$ admits a factorization relative to Γ with partial indices $\kappa_1(s) \geqslant \cdots \geqslant \kappa_n(s)$. It turns out (G. Heinig (1973)) that the partial index tuple $[\kappa_1(s), \ldots, \kappa_n(s)]$ is continuous in s off a finite subset of $\Phi(A)$; the method of proof used in the book is from Azoff, Clancey and Gohberg (1980).

The reviewer has a very high opinion about the book. There is no doubt that the subject concerns a first-rate development in mathematics and its applications. The presentation of the material is excellent and at several places extremely elegant. The main ideas are well explained, the results are clearly stated and appear with full proofs. The book is not heavy in prerequisites;

what the reader has to know to understand the text is explained shortly where it is needed. Misprints occur, but their number is small. The main point is that the book does what it claims to do: It brings together, in a well-organized way, a representative variety of recent results on factorization of matrix functions relative to a contour and systems of singular integral equations. Several results from the Russian literature appear here for the first time in the English language. Many items did not appear in book form before and some (e.g., the notes on shift bases) are new and have not been published elsewhere. It is clear that for a number of years to come the book will be the main reference on factorization along a contour. Further, and probably this was not planned by the authors, the book can easily serve as a text for a one-semester graduate course. At the end of such a course the student not only will be acquainted with the present state of affairs of the subject, but also he will have developed a good understanding of a field which is still active and where further research will take place.

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