BOOK REVIEWS

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Differential forms in algebraic topology, by Raoul Bott and Loring W. Tu, Graduate Texts in Mathematics, Vol. 82, Springer-Verlag, New York, 1982, xiv + 331 pp., $29.80. ISBN 0-3879-0613-4

The once and future theory. A funny thing happened on the way to the future. Once upon a time (1895), when Poincaré recreated analysis situs, he started with what is now called differential topology, using smooth manifolds rather than more general topological spaces. He was well aware of the usefulness of differential forms in a way that was a harbinger of cohomology theory. Indeed the most basic fact, that R^n is cohomologically trivial, is called the Poincaré lemma. Piecing that fact together over the patches of a manifold was more subtle, so subtle that simplicial structures were introduced and dominated throughout most of subsequent history.

There were periods, however, in which differential forms briefly held center stage. Weyl, in his 1911 lectures which became "Die Idee der Riemannischen Fläche" [WY], tells the story for surfaces: differential forms can be integrated and homologous curves in the surface give equal integrals.

It was Elie Cartan who, in 1928, made explicit the formal problems which were resolved by de Rham in 1931. Cartan himself had already made extensive computations of Betti numbers using differential forms, notably for Lie groups. De Rham himself still spoke in terms of Betti numbers and homologous chains, but did relate the exterior product of forms to the intersection of chains. When cohomology became known, according to Garnier and Leray, "one believed cohomology was an entirely new idea, until the day when E. Cartan explained the essence of a page of Henri Poincaré, uncomprehended until then." [p. 184 of Poincaré, *Ouevres*, t.VI].

The thirties saw explosive development of combinatorial methods and, more importantly, the general acceptance of an algebraic point of view. After the disruption of the second world war, Henri Cartan, son of Elie, using the full strength of the point of view of differential graded algebras, made great progress in using differential forms to compute cohomology for homogeneous spaces and certain fibre bundles, and A. Weil used this to provide great insight into the characteristic classes of Chern and Pontrjagin.

Whether these developments could have gone further to handle general fibre spaces is a moot point. What did happen was the breakthrough of Serre's adaptation of the Leray spectral sequence to the singular homology of fibre spaces and the tremendous success of Steenrod's cohomology operations in homotopy theoretic computations — differential forms retired into the background for several decades.

How totally dominant integral and characteristic p techniques were for over two decades is indicated by the treatment of Chern and Pontrjagin classes in Milnor and Stasheff [**MS**]: the definitions in terms of differential forms appear only in the final appendix. Ironically, developments in the real world have at this moment refocussed attention on the form version of characteristic classes: the physics of gauge fields starts with vector potentials (also known as connections) and gauge fields (also known as curvatures) in a whole family of vector bundles over the one point compactification of space time — a vital physical invariant is the instanton number (also known as the Pontrjagin number of the appropriate bundle) computed by physicists precisely in terms of the gauge field as a curvature.

Before this physical development, differential forms found advocates in Whitney [**WH**], Thom [**T**] and Chen [**KTC**], but with little response from the community at large. Quillen's work on rational homotopy theory [**Q**] showed that commutative differential graded algebras (cdga's) gave, in principal, a complete set of rational homotopy invariants, but it took Sullivan's theory of minimal models [**S**] to provide a practical computational tool which led a renaissance of results in characteristic zero cohomology.

Thus, at the present time, three streams have been united or are at least confluent: the classical theory of differential forms on manifolds has found renewed usefulness in mathematical physics, classical homotopy theory has been thoroughly adpated to the world localized at zero and the classical homological theory of commutative local rings has been generalized effectively to the differential graded case. All three of these share the structure of cdga's.

What will the future hold? For the moment, there are plenty of problems and a flood of results to guarantee a generation of research in this area, but already there are attempts to carry this progress away from the rationals and reals. Will the focus shift again, leaving differential forms to lie dormant once more, waiting until decades of integral or mod p results finally grow tired and, once again, turn to differential forms for inspiration?

The future will tell, but for now there is need of texts to instruct a new generation in the integrated point of view, doing homotopy theory including differential forms. For a while the only thing approaching such a text was the set of Florence notes of Friedlander, Griffiths and Morgan, which has lately appeared as *Rational homotopy theory and differential forms* [**GM**] (recently reviewed in Bull. Amer. Math. Soc. (N.S.) **8** (1983), 496 by K.-T. Chen).

This brings us to the subject of our review: *Differential forms in algebraic topology*. Bott and Tu give us an introduction to algebraic topology via differential forms, imbued with the spirit of a master who knew differential forms way back when, yet written from a mature point of view which draws together the separate paths traversed by de Rham theory and homotopy theory. Indeed they assume "an audience with prior exposure to algebraic *or* differential topology". It would be interesting to use Bott and Tu as the text for a first graduate course in algebraic topology; it would certainly be a wonderful supplement to a standard text.

Bott and Tu write with a consistent point of view and a style which is very readable, flowing smoothly from topic to topic. Moreover, the differential forms and the general homotopy theory are well integrated so that the whole is more than the sum of its parts. "Not intended to be foundational", the book presents most key ideas, at least in sketch form, from scratch, but does not hesitate to quote as needed, without proof, major results of a technical nature, e.g., Sard's Theorem, Whitney's Embedding Theorem and the Morse Lemma on the form of a nondegenerate critical point.

Comments on the arrangement of topics are in order since the topics are not, or at least are not in the order of any of the standard texts in algebraic topology.

Chapter I, "De Rham theory", begins with $\Omega^*(R^n)$, the C^∞-forms on R^n with the exterior derivative and then regards $\Omega^*(\ )$ as a functor (with respect to smooth maps) to the category of commutative differential graded algebras. This functor is then extended to differential manifolds via atlases of charts, i.e. local coordinate systems. The piecing together of forms is handled explicitly by the Mayer-Vietoris sequence of forms

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0.$$

Stokes' theorem for an oriented manifold with boundary is then proved, first for R^n and then in general. Homotopy of maps and chain maps is introduced to prove the Poincaré lemma. Thus all the tools are ready for major results on manifolds with finite good covers (i.e. open covers such that all finite intersections are diffeomorphic to R^n; throughout the book, these good covers will play a key role in the development). Poincaré duality, the Künneth formula, Leray-Hirsch for fibre bundles with fibre totally noncohomologous to zero, and the Thom isomorphism for sphere bundles follow in quick succession. The last of these is defined using integration along the fibre (after all, differential forms are what go after integral signs).

Chapter II, "The Čech-de Rham Complex", reformulates the Mayer-Vietoris sequence as the Mayer-Vietoris complex of an open cover. That this double complex computes the de Rham cohomolgy is (re)proved first for a covering $U \cup V$ and then for a general covering by a specific zig-zag argument, displayed on the first quadrant lattice. If the cover is good, the double complex also computes the Čech cohomology and the de Rham theorem follows. Rather than do this via a spectral sequence, the specific argument here provides excellent motivation for the generalities of spectral sequences presented in the next chapter.

Chapter III, "Spectral Sequences and Applications", is rather misleadingly titled since it also includes singular homology theory and a review of homotopy theory from the definition of homotopy groups through CW-structures (given by Morse theory) and Postnikov systems. The applications then include the cohomology of $K(Z,3)$ and $\Pi_i(S^3)$ for $i \leqslant 5$. The treatment of the Hopf invariant for the Hopf map of S^3 to S^2 is a particularly fine example of the combination of integral methods (e.g., the linking number definition) and form methods (e.g., the functional exterior product definition).

The treatment of spectral sequences emphasizes the one for fibrations, but the examples proceed from the trivial to the subtle; an excellent way to learn.

The final section of the chapter is an extremely brief introduction to rational homotopy theory, meaning here Sullivan's theory of minimal models as applied to $\Pi_*(X) \otimes Q$ as the space of generators of the minimal models. There is a forbidding reference to the need for technical details, apparently written without knowledge of Halperin's Lille notes [H]. More surprising is the brevity of the examples given: $\pi_*(S^2 \vee S^2) \otimes Q$ is given through π_6 with no mention of the graded Lie algebra structure on all of $\pi_* \otimes Q$ and the striking results on $\pi_* \otimes Q$ for Lie groups are not even alluded to.

Chapter IV, "Characteristic Classes", is meant as a capstone, giving one major application of the techniques so far developed, both specific differential forms and general algebraic homotopy theoretic manipulations. Chern classes are introduced specifically for line bundles in terms of transition functions and then, following Grothendieck, are developed in general by using the splitting principle for the associated projective bundle.

There is also a discussion of the finite Grassmannians $G_K(\mathbf{C}^n)$ as classifying spaces, specifically presented for vector bundles over compact finite-dimensional manifolds. Finally, formulas for Chern classes in terms of forms and transition functions are given rather telegraphically, deliberately avoiding the language of connections and curvatures.

Overall, the presentation, even the notation and use of diagrams, makes the material easily accessible; the obvious shortcomings are largely the result of stopping short after having, as the authors intended, "open(ed) some of the doors to the formidable edifice of algebraic topology". The interested student or proselytizing teacher can readily move on to explore the vistas beyond those open doors.

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Finite groups. II, by B. Huppert and N. Blackburn, Grundlehren der Mathematischen Wissenschaften Series 242, Springer-Verlag, Berlin and New York, 1982, xiii + 531 pp., $68.00. ISBN 0-3871-0632-4

Finite groups. III, by B. Huppert and N. Blackburn, Grundlehren der Mathematischen Wissenschaften Series 243, Springer-Verlag, Berlin and New York, 1982, viii + 454 pp., $59.00. ISBN 0-3871-0633-2

Group theory. I, by Michio Suzuki, Grundlehren der Mathematischen Wissenschaften Series 247, Springer-Verlag, Berlin and New York, 1982, xii + 434 pp., $48.00. ISBN 3-5401-0915-3

After three decades of intensive research by hundreds of group theorists, the century old problem of the classification of the finite simple groups has been solved and the whole field has been drastically changed. A few years ago the one focus of attention was the program for the classification; now there are many active areas including the study of the connections between groups and geometries, sporadic groups and, especially, the representation theory. A spate of books on finite groups, of different breadths and on a variety of topics, has appeared, and it is a good time for this to happen. Moreover, the classification means that the view of the subject is quite different; even the most elementary treatment of groups should be modified, as we now know that all finite groups are made up of groups which, for the most part, are imitations of Lie groups using finite fields instead of the reals and complexes. The typical example of a finite group is $GL(n, q)$, the general linear group of n dimensions over the field with q elements. The student who is introduced to the subject with other examples is being completely misled.

The volume by Suzuki can stand on its own as an introduction to group theory, emphasizing finite groups, even though it is preparation for the second volume which will deal largely with simple groups and methods of studying them. The style is detailed and leisurely with many examples and exercises. The first chapter is devoted to the rudiments of the subject but closes, very appropriately, with a study of the general linear groups over arbitrary fields. The third chapter is also given mainly to the most important examples of