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APPROXIMATION BY POLYNOMIALS IN TWO DIFFEOMORPHISMS

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We denote by $\mathbf{C}$ the complex plane. If f and g are complex-valued functions on a set S, then $\mathbf{C}[f,g]$ denotes the algebra of polynomials in f and g, with complex coefficients, regarded as functions on S.

THEOREM. *Let $1 \leq k \in Z$, and let f and g be C^k diffeomorphisms of $\mathbf{C}$ into $\mathbf{C}$, having opposite degrees. Then $\mathbf{C}[f,g]$ is dense in the Fréchet space $C^k(\mathbf{C})$, i.e., given $h \in C^k(\mathbf{C})$, and $X \subset \mathbf{C}$ compact, there is a sequence $h_n \in \mathbf{C}[f,g]$ such that h_n and its derivatives up to order k tend to h and its derivatives, uniformly on X.*

In case $f(z) = z$ and $g(z) = \bar{z}$, the Theorem reduces to a result of Weierstrass. Since each diffeomorphism of the closed unit disc D into $\mathbf{C}$ extends to a diffeomorphism of $\mathbf{C}$ into $\mathbf{C}$, we deduce the following.

COROLLARY. *Let f and g be C^1 diffeomorphisms of D into $\mathbf{C}$, having opposite degrees. Then $\mathbf{C}[f,g]$ is dense in $C(D)$.*

This settles an old chestnut in the field of uniform algebras. It remains open whether the Corollary works for $k = 0$, i.e., for all pairs of homeomorphisms of opposite degrees.

PROOF OF THEOREM. Without loss of generality, we may take $g = z$, because the chain rule for $D^j(h \circ g)$ is linear in h and involves only $D^i h$ and $D^i g$ for $0 \leq i \leq j$.

Since f has degree -1, we deduce that $|f_{\bar{z}}| > |f_z|$ on $\mathbf{C}$. In particular, $f_{\bar{z}} \neq 0$, so the graph $G = \{(z, f(z)) \in \mathbf{C}^2 : z \in \mathbf{C}\}$, which is a C^k submanifold of $\mathbf{C}^2$, has no complex tangents. By the Range-Siu theorem [2], $C^k(G)$ is the closure of the space $\mathcal{O}(G)$ of all functions holomorphic in a neighbourhood of G. If we can show that G has an exhaustion by polynomially-convex compact sets, then by the functional calculus [4, Chapter 8], it will follow that $\mathbf{C}[z,w]$ is dense in $\mathcal{O}(G)$, and hence in $C^k(G)$; since $z \mapsto (z,f)$ is a C^k diffeomorphism of $\mathbf{C} \to G$, this will imply that $\mathbf{C}[z,f]$ is dense in $C^k(\mathbf{C})$. Thus it suffices to show that $X = \{(z, f(z)) : z \in K\}$ is polynomially-convex whenever $K \subset \mathbf{C}$ is a closed disc.

Fix a closed disc $K \subset \mathbf{C}$. By modifying f off K, if need be, we may assume f maps $\mathbf{C}$ *onto* $\mathbf{C}$, that Df and Df^{-1} are bounded and uniformly continuous, and that $|f_{\bar{z}}|$ and $1 - |f_z/f_{\bar{z}}|$ are bounded away from zero. We need two lemmas, which are essentially classical results of Wermer.

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LEMMA 1. *There exists a constant* $\lambda_1 > 0$ *such that*

$$(z-a)(f(z)-f(a))+\lambda f_{\bar{z}}(a)$$

is nonzero whenever $0 < \lambda < \lambda_1$, $a \in \mathbf{C}$, *and* $z \in \mathbf{C}$.

PROOF. Pick $\delta > 0$ such that the modulus of continuity $\omega(\delta)$ of Df at δ is less than half $(\inf |f_{\bar{z}}|)(1-\sup_{\mathbf{C}} |f_z/f_{\bar{z}}|$. Applying the mean value theorem to the real and imaginary parts of f we deduce that for $0 < |z-a| < \delta$, the value $f(z)-f(a)$ differs from $f_{\bar{z}}(a)\overline{(z-a)}+f_z(a)(z-a)$ by less than $2\omega(\delta)|z-a|$. Thus

$$\operatorname{Re}\frac{(z-a)(f(z)-f(a))}{f_{\bar{z}}(a)} \geq 0$$

whenever $|z-a| < \delta$. But for $|z-a| \geq \delta$,

$$\left|\frac{(z-a)(f(z)-f(a))}{f_{\bar{z}}(a)}\right| \geq \frac{\delta^2(\sup|Df^{-1}|)^{-1}}{\inf|f_{\bar{z}}|}.$$

Denoting the right-hand side by λ_1, we see that $(z-a)(f(z)-f(a))/f_{\bar{z}}(a)$ omits $\{-\lambda\colon 0 < \lambda < \lambda_1\}$, for all a and z, so the lemma is proved.

Let us denote the uniform closure of $\mathbf{C}[z,f]$ in $C(K)$ by A.

LEMMA 2. *Suppose that for each* $a \in K$, *there exists a sequence* $\lambda_n \downarrow 0$ *such that* $(z-a)(f(z)-f(a))+\lambda_n f_{\bar{z}}(a)$ *is invertible in* A. *Then* $A = C(K)$.

PROOF. Briefly, let μ be a measure on K, annihilating A. It suffices to show that the Cauchy transform $\hat{\mu}(a) = \int d\mu(\varsigma)/\varsigma - a$ vanishes at every point $a \in K$ at which the Newtonian potential $\int d|\mu|(\varsigma)/|\varsigma - a|$ is finite. But the hypothesis, together with Lemma 1, yields a sequence $f_n \in A$ such that $f_n \to (z-a)^{-1}$, pointwise on $K \sim \{a\}$, and $|f_n(z)| \leq \text{const } |z-a|^{-1}$. Thus the dominated convergence theorem yields the desired result.

We remark that the hypothesis of Lemma 2 can be weakened to "almost all $a \in K$".

CONCLUSION OF PROOF OF THEOREM. Suppose X is not polynomially-convex. Then $A \neq C(K)$, so by Lemma 2, there exists $a \in K$ and $\lambda_2 > 0$ such that for every λ with $0 < \lambda < \lambda_2$, the polynomial $(z-a)(w-f(a))+\lambda f_{\bar{z}}(a)$ has a zero somewhere on the polynomially-convex hull of X. Fix λ, with $0 < \lambda < \min\{\lambda_1, \lambda_2\}$. Then the family of algebraic curves

$$(z-a-t)(w-f(a+t))+\lambda f_{\bar{z}}(a+t) = 0 \qquad (0 \leq t < \infty)$$

is a curve of algebraic hypersurfaces which meets the hull of X, does not meet X (by Lemma 1), and goes to the hyperplane at infinity (since f maps *onto* $\mathbf{C}$, and $f_{\bar{z}}$ is bounded). This contradicts Oka's characterization of polynomial hulls, as given in [**3**, (1.2), p. 263]. Thus X is polynomially-convex, and we are done.

We remark that minor modifications to the foregoing proof permit us to strengthen the Corollary, as follows:

Let f be an orientation-reversing homeomorphism of $\mathbf{C}$ into $\mathbf{C}$, which is locally C^1 and noncritical off a closed set E, having area zero and not separating the plane. Then $\mathbf{C}[z,f]$ is dense in $C(\mathbf{C})$.

Also, for any compact set X in $\mathbf{C}$ and for $0 < \alpha < 1$, suppose $\mathrm{Lip}(\alpha, X)$ denotes the space of bounded functions g of X into $\mathbf{C}$ such that for some $K > 0$, $|g(z) - g(w)| \leq K|z-w|^\alpha$ for all z, $w \in X$ with norm $\sup|g| + \text{Least } K$ and suppose $\mathrm{lip}(\alpha, X)$ denotes those functions $g \in \mathrm{Lip}(\alpha, X)$ such that, given $\epsilon > 0$, there exists $\delta > 0$ such that $|g(z) - g(w)| \leq \epsilon|z-w|^\alpha$ whenever z and w satisfy $|z - w| < \delta$. In view of the results given in [**1**, p. 227], the conclusion of the above remark implies $\mathbf{C}[z, f]$ is dense in $\mathrm{lip}(\alpha, X)$ for any compact set X in $\mathbf{C}$.

Finally, we remark that the Theorem of this paper is sharp in the sense that one critical point destroys it.

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