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FIXED POINT THEORY AND NONLINEAR PROBLEMS

BY FELIX E. BROWDER

Introduction. Among the most original and far-reaching of the contributions made by Henri Poincaré to mathematics was his introduction of the use of topological or "qualitative" methods in the study of nonlinear problems in analysis. His starting point was the study of the differential equations of celestial mechanics, and in particular of their periodic solutions. His work on this topic began with his thesis in 1879, and was developed in detail in his great three-volume work, *Méthodes nouvelles de la méchanique céleste*, which appeared in the early 1890s and summarized his many memoirs of the intervening period. It continued until his memoir shortly before his death in 1912 in which he put forward the unproved fixed point result usually referred to as "Poincaré's last geometric theorem".

The ideas introduced by Poincaré include the use of fixed point theorems, the continuation method, and the general concept of global analysis. The writer's acquaintance with Poincaré's influence came through contact with Solomon Lefschetz and Marston Morse, both of whom were very explicit as to the role of Poincaré as an initiator in this direction of mathematical development. In 1934, in the Foreword to his Colloquium volume on *The calculus of variations in the large*. Morse had put this forward very forcefully in the first paragraph:

> "For several years the research of the writer has been oriented by a conception of what might be termed macro-analysis. It seems probable to the author that many of the objectively important problems in mathematical physics, geometry, and analysis cannot be solved without radical additions to the methods of what is now strictly regarded as pure analysis.

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> Any problem which is nonlinear in character, which involves more than one coordinate system or more than one variable, or whose structure is initially defined in the large, is likely to require considerations of topology and group theory in order to arrive at its meaning and its solution. In the solution of such problems classical analysis will frequently appear as an instrument in the small, integrated over the whole problem with the aid of group theory or topology. Such conceptions are not due to the author. It will be sufficient to say that Henri Poincaré was among the first to have a conscious theory of macro-analysis, and of all mathematicians was doubtless the one who most effectively put such a theory into practice."

I know off-hand of no such written testimony by Lefschetz, though his advice in conversation included the recommendation that everyone strongly interested in nonlinear problems in analysis should read through the Collected Works of Poincaré. How seriously this advice was meant is relatively difficult to determine after all this time. It always seemed to me that this was advice meant to frighten, not to encourage effort in this area, somewhat like the recommendation of André Weil that every aspiring algebraic number theorist should read through Hilbert's *Zahl-Bericht* once a year. However, we can associate Lefschetz with the analysis offered by Liapounoff in the classical memoir on stability theory of 1892 which Lefschetz caused to be republished in the Annals of Mathematics Studies in 1947. In this work, which was translated and published in French in 1907 under the title *Problème général de la stabilité du mouvement*, Liapounoff wrote

> "L'essai unique, autant que je sache, de solution rigoreuse de la question appartient à M. Poincaré, qui dans le Mémoire remarquable sous bien des rapports *Sur les courbes définies par les equations différentielles* et, en particulier, dans les deux dernières Parties, considère des questions de stabilité relatives au cas d'équations différentielles du second ordre et s'arrète aussi à quelques questions voisines, se rapportant à des systèmes du troisième ordre.
>
> Bien que M. Poincaré se borne a des cas très particuliers, les méthodes dont il se sert permettent des applications beaucoup plus générales et peuvent encore conduire à beaucoup de nouveaux résultats. C'est ce qu'on verra par ce qui va suivre, car, dans une grande partie des mes recherches, je me suis guidé par les idées développés dans le Mémoire cité."

Let me conclude this kind of textual testimony by one more quotation, this time from Jurgen Moser's Annals of Mathematics Study, *Stable and random motions in dynamical systems*, published in 1973. In the Introduction (page 4), he writes

> "However, the mathematical difficulties connected with this field, inspired more and more the study of basic theoretical problems leading to the development of new mathematical tools. The main influence in this direction came from Poincaré who started a number of new lines of thought, like the qualitative theory of differential equations, the quest for the topology of the energy manifold, he formulated and proved fixed point theorems to establish the existence of periodic solutions. In this connection one may recall that the fixed point theorem by P. Bohl and Poincaré also had its origin in this field, to which Bohl devoted his entire mathematical life."

The topic to which the present paper is devoted, degree theory for nonlinear mappings, is one to which Poincaré made an early and important contribution. His first results appear in a note in the Comptes Rendus [32] and a more detailed development [33] published in the Bulletin Astronomique, both in 1883 only four years after his Thesis. In this pair of papers, Poincaré announces the following result (in my translation): "Let $\xi_1, \xi_2, \ldots, \xi_n$ be n continuous functions of n variables $x_1, x_2, \ldots, x_n$: the variable x_i is subjected to vary between the limits $+a_i$ and $-a_i$. Let us suppose that for $x_i = a_i$, ξ_i is constantly positive, and that for $x_i = -a_i$ constantly negative; I say that there will exist a system of values of x for which all the ξ's vanish."

For the proof, he refers to the celebrated paper by Kronecker in 1869 [25] which began the theory of the topological degree of a mapping. (For a detailed analysis of this paper and of the subsequent literature on the Kronecker index, see the recent historical paper by Siegberg [35]. The reference to the 1883 papers by Poincaré, I owe to Mahwin [27]. Most accounts of Poincaré's role begin with his paper [34] in 1886.) No account of the proof is given in the 1883 papers, but in 1886, Poincaré [34] published the argument on the continuation invariance of the index which is the basis for the proof. This is the memoir cited by Liapounoff, and like the 1883 announcements, applies what amounts to a fixed point argument to prove the existence of periodic solutions of a system of ordinary differential equations.

We can describe the basic idea in the following general terms: Suppose we are given a system of differential equations

$$\frac{du}{dt}(t) = T(u(t))$$

where the right-hand side is independent of time t. Suppose this system has the property that for each initial value u_0, the system has one and only one solution defined over an interval $[0, T]$. Then the problem of finding a periodic solution $u(t)$ of period T is equivalent to finding an initial value u_0 such that the solution $u(t)$ with that given initial value has $u(T) = u_0$. If we set

$$S(t)u_0 = u(t),$$

this means finding a fixed point of the mapping $S(T)$.

The continuation method, a favorite technique of Poincaré, consists of imbedding the problem in a one-parameter family of problems depending upon an auxiliary parameter s and considering the solvability of the problem as s varies. The link between these two devices is the index or topological degree, which assures under appropriate hypotheses that the solvability of a family of problems is invariant under continuous perturbations. This continuity or homotopy invariance is the decisive property of the topological degree of mapping. Thus, in the 1880s, Poincaré was effectively using the basic properties of the topological degree, usually associated with L. E. J. Brouwer [7] in 1912. The result stated by Poincaré has come to be known as the theorem of Miranda because it was proved by C. Miranda [**28**] in 1940, who showed that it was equivalent to the Brouwer fixed point theorem. (It had earlier been applied by Michal Golomb [**21**] in 1935, who observed that its proof was implicit in another paper [**6**] by Brouwer in 1911.)

In the present paper, we give a simple exposition of the classical theory of the degree of a mapping as given by Kronecker and Brouwer, and of its extension by Leray and Schauder [**26**] in 1934 to mappings in infinite-dimensional Banach spaces of the form $I - g$, with g compact. We discuss in detail the existence and uniqueness of these degrees as defined by the additional properties of additivity, homotopy invariance, and normalization. We also present a self-contained exposition of the recent extension given by the writer [**11–14**] of these existence and uniqueness results for the degree functions for nonlinear mappings of monotone type from a reflexive Banach space X to its conjugate space X^*. We show how such mappings arise from the combination of the ideas of fixed point theory and the somewhat different circle of ideas associated with the direct method of the calculus of variations. The concept of degree of mapping in all these forms is one of the most effective tools for studying the properties of existence and multiplicity of solutions of nonlinear equations.

1. Degree theory in the finite-dimensional case. In the present section, we wish to develop as intuitively as possible, the fundamental properties of the classical topological degree as they were explicitly formulated by Brouwer in 1912 (and implicitly used by Poincaré, Bohl, and others (e.g. Picard [**31**]) on the basis of Kronecker's earlier results about the index).

Let us begin with some conventions on notation. We shall consider mappings with domain in a topological space X and with values lying in a topological space

Y. If G is an open subset of X, we denote by $\mathrm{cl}(G)$ its closure in X, and by ∂G, its boundary in X. Thus $f\colon \mathrm{cl}(G) \to Y$ will be the prototype of the maps for which a degree function is to be defined. If $f\colon \mathrm{cl}(G_0) \to Y$ is such a map and G is an open subset of G_0, then by f_G we mean the restriction of the given map f to $\mathrm{cl}(G)$.

THEOREM 1. *Let $X = R^n = Y$ for a given positive integer n. For bounded open subsets G of X, consider continuous mappings $f\colon \mathrm{cl}(G) \to Y$, and points y_0 in Y such that y_0 does not lie in $f(\partial G)$. Then to each such triple (f, G, y_0), there corresponds an integer $d(f, G, y_0)$ having the following properties*:

(a) *If $d(f, G, y_0) \neq 0$, then $y_0 \in f(G)$. If f_0 is the identity map of X onto Y, then for every bounded open set G and $y_0 \in G$, we have*

$$d(f_{0,G}, G, y_0) = +1.$$

(b) (*Additivity*). *If $f\colon \mathrm{cl}(G) \to Y$ is a continuous map with G a bounded open set in X, and G_1 and G_2 are a pair of disjoint open subsets of G such that*

$$y_0 \notin f(\mathrm{cl}(G)\setminus(G_1 \cup G_2))$$

then

$$d(f, G, y_0) = d(f_{G_1}, G_1, y_0) + d(f_{G_2}, G_2, y_0).$$

(c) (*Invariance under homotopy*). *Let G be a bounded open set in X, and consider a continuous homotopy $\{f_t\colon 0 \leq t \leq 1\}$ of maps of $\mathrm{cl}(G)$ into Y. Let $\{y_t\colon 0 \leq t \leq 1\}$ be a continuous curve in Y such that $y_t \notin f_t(\partial G)$ for any t in $[0, 1]$. Then $d(f_t, G, y_t)$ is constant in t on $[0, 1]$.*

THEOREM 2. *The degree function $d(f, G, y_0)$ is uniquely determined by the three conditions of Theorem 1.*

Theorem 1 is an appropriately formalized version of the properties of the classical Brouwer degree. Theorem 2 contains an observation made independently in 1972 and 1973 by Fuhrer [**20**] and Amann and Weiss [**2**], respectively.

What is this degree function which we describe in Theorems 1 and 2? Intuitively, it is intended to be an algebraic count of the number of solutions x in G for the equation $f(x) = y_0$. We speak of algebraic count because (as we specify more concretely in a moment) some solutions are counted positively, others negatively. The paradigmatic case is that of a mapping f of class C^1 with only regular points x as solutions of the equation $f(x) = y_0$, i.e. at each solution $f'(x)$ is a nonsingular linear transformation of R^n. The number of such solutions is then finite, and we describe such solutions as positive if $f'(x)$ preserves orientation and negative if it reverses orientation. Then $d(f, G, y_0)$ equals the number of positive solutions minus the number of negative solutions. Thus defined, the degree function for such mappings is certainly at least an integer. It is also clearly additive and satisfies the normalization condition. The main problem is to show that is extendable to all mappings and is homotopy invariant.

One way of establishing these facts is to identify $d(f, G, y_0)$ for such maps with an integral of the type studied by Kronecker. To do this, let us assume first that G

has a C^1 boundary $S = \partial G$ which we orient in an appropriate way so as to satisfy the conditions for Stokes theorem:

$$\int_G d\alpha = \int_S \alpha$$

for any C^1 $(n-1)$-form α on cl(G). We shall describe an $(n-1)$-form ω on $R^n \backslash \{0\}$ in terms of the fundamental solution $e_n(r)$ of radial type for the Laplacian on R^n. Thus for $n > 2$, $e_n(r) = c_n r^{2-n}$, while for $n = 2$, $e_2(r) = c_2 \ln(1/r)$. Thus, setting $\Delta = \sum_{j=1}^n (\partial/\partial x_j)^2$, we have

$$\Delta(e_n) = \delta,$$

where δ is the Dirac delta function at 0. We set

$$\omega = \sum_{j=1}^n (-1)^j \frac{\partial}{\partial x_j}(e_n(r))\, d\hat{x}_j$$

with $r = (\sum_{j=1}^n x_\gamma^2)^{1/2}$ and $d\hat{x}_j = dx_1 \cdots \not{d}\!\not{x}_j \cdots dx_n$. Then

$$\int_{\partial G} \omega = \begin{cases} +1 & \text{if } 0 \in G, \\ 0 & \text{if } 0 \notin \text{cl}(G). \end{cases}$$

Using the form $f_*(\omega)$ on S induced by the map f of S into R^n, we assert that for maps f having only regular points in $f^{-1}(0)$

$$d(f, G, 0) = \int_{\partial G} f_*(\omega).$$

We need only observe that on any domain G_0 for which f has no zeroes, $df_*(\omega) = f_*(d\omega) = 0$. Hence

$$\int_{\partial G_0} f_*(\omega) = 0$$

for such G_0. We can take G_0 to be the complement in G of the union of a family of small balls around the various points of the finite set $\{x_1, \ldots, x_r\} = f^{-1}(0)$. As the balls are taken smaller and smaller (if B_j is the ball around the jth point x_j)

$$\int_G f_*(\omega) = \sum_{j=1}^r \int_{\partial B_j} f_*(\omega)$$

while

$$\int_{\partial B_j} f_*(\omega) - \int_{\partial B_j} L_{j*}(\omega) \to 0$$

where L_j is the linear map $f'(x_j)$. Finally,

$$\int_{\partial B_j} L_{j*}(\omega) = \text{sgn}\big(\det(L_j)\big).$$

If we combine these facts, we identify $d(f, G, 0)$ with the corresponding integral.

A similar representation holds for $d(f, G, y_0)$ if we note that $d(f, g, y_0) = d(f^{(y_0)}, G, 0)$, where $f^{(y_0)}(x) = f(x) - y_0$ for all x in cl(G).

The point of identifying $d(f, G, y_0)$ with an integral is that the integral makes sense for any C^1 mapping f, not just for the ones with regular points in $f^{-1}(y_0)$, provided only that $f(S)$ does not contain y_0. Moreover, it is continuous under C^1 deformations of the mapping f, under the sole restriction that during the deformation y_0 never appears in $f(\partial G)$.

We now apply the theorem of Sard-Morse in the case of maps of R^n into R^n to approximate f in the C^1 topology by a map g having only regular points in its zero set with $\| f - g \|_{C^1(\mathrm{cl}(G))} < \varepsilon$. It follows that for any f and the corresponding g, the integrals for f and g are very close. Since g is one of the paradigmatic maps we were treating earlier, $d(g, G, 0)$ is an integer. Hence, the integral for f defining $d(f, G, 0)$ being arbitrarily close to an integer, must itself be an integer. However, an integer-valued function of a parameter which is continuous in that parameter must be constant. Hence $d(f_s, G, y_0)$ is constant under any C^1 deformation during which it remains defined.

Thus, we have a degree function $d(f, G, y_0)$ defined for f of class C^1 and G having a C^1 boundary. We now proceed to remove these restrictions on the smoothness of f and G.

Let G be any bounded open subset of R^n and consider a continuous map f: $\mathrm{cl}(G) \to R^n$ which is of class C^2 in G and such that $0 \notin f(\partial G)$. Then we can find $\gamma > 0$ such that on the γ-neighborhood of ∂G,

$$|f(x)| > \gamma.$$

We consider a C^1 function $\varphi(r)$ which is 1 for $r \geqslant \gamma$, 0 for $r < \gamma/2$. Set

$$\tilde{\omega} = \varphi(r)\omega.$$

For any open subset G_1 of G which has a C^1 boundary and contains the complement in G of the γ-neighborhood of ∂G, the boundary G_1 is contained in the γ-neighborhood of ∂G. Hence

$$d(f_{G_1}, G_1, 0) = \int_{\partial G_1} f_*(\omega) = \int_{\partial G_1} f_*(\tilde{\omega}).$$

However, the form $f_*(\tilde{\omega})$ is of class C^1 in $\mathrm{cl}(G_1)$. Hence

$$\int_{\partial G_1} f_*(\tilde{\omega}) = \int_{G_1} df_*(\tilde{\omega}) = \int_{G_1} f_*(d\tilde{\omega}) = \int_{G_1} f_*(d\varphi \wedge \omega).$$

Since

$$f_*(d\varphi \wedge \omega) = 0$$

on the γ-neighborhood of ∂G, we see that the resulting integral is independent of the choice of G_1 and can be expressed as an integral over $G \backslash N_\gamma(\partial G)$, with $N_\gamma(\partial G)$ the γ-neighborhood.

Finally, suppose f: $\mathrm{cl}(G) \to R^n$ is merely continuous. Then we can approximate it arbitrarily closely in the C^0-norm by C^2 maps g. If h is another such approximation, then so is

$$g_t = (1 - t)g + th$$

for any t in [0, 1], and for any t, $g_t(x) \neq 0$ for x in ∂G. Hence

$$d(g_t, G, 0)$$

is independent of t in [0, 1] since $\{g_t\}$ is a C^1 homotopy of C^1 maps. We define

$$d(f, G, 0) = d(g, G, 0),$$

and more generally, $d(f, G, y_0) = d(g, G, y_0)$ for such approximations g.

The degree function thus defined is immediately seen to satisfy the three conditions of Theorem 1.

For the normalization conditions, this follows immediately for the maps with only regular solutions of $f(x) = y_0$ by the definition of the degree function for such maps. (Moreover, the identity map is immediately seen to have the appropriate value for its degree function.) For more general maps f, our process of definition yields a map g with only regular solutions such that g is close to f in the C^0-norm and

$$d(f, G, y_0) = d(g, G, y_0).$$

Hence, if $\| f - g \|_{C^0} < \varepsilon$, since $d(f, G, y_0) \neq 0$, we see that g must have y_0-points in G, so that there exist points in cl(G) such that $\| f(x) - y_0 \| < \varepsilon$ for any $\varepsilon > 0$. Since $f(\text{cl}(G))$ is closed, it follows that $y_0 \in f(\text{cl}(G))$. Since y_0 does not lie in $f(\partial G)$ by assumption, y_0 must belong to $f(G)$. Additivity on domain follows immediately by the corresponding additivity for maps with regular y_0-points. Finally, given a continuous homotopy $\{f_t\}$ in the C^0 topology and a continuous curve $\{y_t\}$ in R^n, we can consider $\{f_t - y_t\}$, and show that $d(f_t, G, 0)$ is constant after the modification of the f_t. We can find $\gamma > 0$ such that $|f_t(x)| > \gamma$ for x on ∂G. We now choose a sequence $0 = t_0 < t_1 < \cdots < t_r = 1$ such that $\| f_{t_j} - f_{t_{j+1}} \|_{C^0} < \gamma/2$. For each j, choose a C^1-map g_j: cl(G) $\to R^n$ such that

$$\| f_{t_j} - g_j \|_{C^0(\text{cl}(G))} < \gamma/2.$$

Then for each j, we have

$$d(f_{t_j}, G, 0) = d(g_j, G, 0),$$

and it suffices to show that for each j,

$$d(g_j, G, 0) = d(g_{j+1}, G, 0).$$

We consider the linear homotopy

$$h_t = (1 - t)g_j + tg_{j+1}$$

between g_j and g_{j+1}. This homotopy is of class C^1, so that to obtain the constancy of the degree under the homotopy, it suffices to show that for all x in ∂G and t in [0, 1], $h_t(x) \neq 0$. Both points $g_j(x)$ and $g_{j+1}(x)$ are contained in the open ball of radius γ about the point $f_{t_j}(x)$, and hence so is their convex linear combination $h_t(x)$. Since this ball does not contain 0, $h_t(x) \neq 0$.

REMARK. Analytical proofs for the existence and properties of the degree function have been given a number of times in the literature in the last forty years, for the first time explicitly by Nagumo in 1941. A recent version with bibliographical references is that of Fenske [**19**].

We now observe that the argument given above, together with one further observation, yields the proof of Theorem 2. Suppose that we are given a degree function $d_1(f, G, y_0)$ which satisfies the three conditions of Theorem 1. We wish to identify it with the degree function d already constructed. We may approximate f as closely as we please in the C^0-norm by a map g of class C^1 which has only finitely many points $\{x_1,\dots,x_r\}$ in G at which $f(x) = y_0$, and all of these points are regular points. Since f and g are close, the linear homotopy

$$h_t = (1 - t)f + tg$$

is permissible in that it produces no points x on ∂G such that $h_t(x) = y_0$ for any t in $[0, 1]$. By homotopy invariance of the degrees,

$$d_1(f, G, y_0) = d_1(g, G, y_0), \qquad d(f, G, y_0) = d(g, G, y_0).$$

Hence, we need only show the equality of the two degrees on such mappings g. Moreover, using the additivity on domain, we can replace G by the union of small balls B_j around the various points x_j. It thus suffices to take $G = B$ where B is the ball of radius ε about the point x_0 which is the unique solution of $g(x) = y_0$ in B. If $\varepsilon > 0$ is sufficiently small and if L is the derivative of g at x_0, the homotopy between g and g_0 with $g_0(x) = y_0 + L(x - x_0)$ given by

$$k_t = (1 - t)g + tg_0$$

has the property that it produces no y_0 points on ∂B. Hence, we need only verify that

$$d(g_0, B, y_0) = d_1(g_0, B, y_0).$$

Furthermore, if we deform L through the nonsingular linear mappings of R^n, both degrees remain constant. Since the nonsingular linear mappings of L have two path components, the maps with positive determinant and the maps with negative determinant, it suffices to consider a single map in each class. For the mappings with positive determinant (the orientation-preserving maps), we choose the identity map I, and the equality of the two degrees follows from the normalization condition. In the other case, we choose the linear mapping L given by

$$Lx_1 = -x_1, \qquad Lx_j = x_j \quad \text{for } j \geqslant 2.$$

To handle this last case, let us observe that it really amounts to considering the same problem in the one-dimensional case, $n = 1$. Indeed, let B_{n-1} be the unit ball in R^{n-1} and for any mapping f_1: $\mathrm{cl}(G_1) \to R^1$ with G_1 a bounded open subset in R^1, let us define

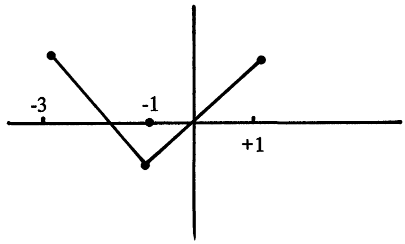
$$G_2 = x_0 + G_1 \times B_{n-1}, \qquad f_2(x) = y_0 + h(x), \qquad h(x) = (f_1(x_1), x_2,\dots,x_n).$$

We define two degrees d and d_2 on the one-dimensional maps by setting

$$d(f_1, G_1, u) = d(f_2, G_2, y_0 + u), \qquad d_2(f_1, G_1, u) = d_1(f_2, G_2, y_0 + u)$$

where u is considered as a vector $(u,0,\dots,0)$. The degree function d is the conventional one on R^1; we must show it identical to the degree function d_2

defined in terms of d_1. For $n = 1$, the problem is best resolved by the following picture:



The function h whose graph is pictured on the interval $[-3, +1]$ has two zeros at 0 and at (-2). Its total degree over $G_0 = (-3, +1)$ with respect to 0 must be 0, since it is linearly homotopic to the constant function $h_0(x) \equiv 1$ without zeroes on the boundary. Since

$$d_2(h, G_0, 0) = d_2(h, G_1, 0) + d_2(h, G_2, 0)$$

where $G_1 = (-3, -1)$ and $G_2 = (-1, +1)$ and since $d_2(h, G_2, 0) = +1$, $d_2(h, G, 0) = 0$, we have

$$d_2(h, G_1, 0) = -1$$

which verifies our assertion for L.

This completes the proof of Theorem 2.

2. General degree theories and their elementary properties. Using the result of §1 as a model, we may formulate the problem of constructing more general degree theories in the following terms:

DEFINITION 1. *We are given a domain space X and a range space Y, both topological spaces.*

We are given a class O of open subsets G of X.

For each G in O, we consider a family of maps f: $\mathrm{cl}(G) \to Y$; the collection of all such maps for the various G of O, we call F, the family of maps over which the degree theory is to be defined.

For each G in O, we consider a family of homotopies $\{f_t: 0 \leq t \leq 1\}$ of maps in F, all having the common domain $\mathrm{cl}(G)$; the total collection of all such homotopies for the various G in O, we call H, the class of permissible homotopies for the degree theory.

Then by a degree theory over the class F which is invariant with respect to the homotopies of H and which is normalized by a given map f_0 of X into Y, we mean: For each y_0 in Y and for each f in F, f: $\mathrm{cl}(G) \to Y$ for which $y_0 \notin f(\partial G)$, an integer $d(f, G, y_0)$ is to be prescribed and the prescription is to satisfy the following three conditions:

(a) (*Normalization*) *If $d(f, G, y_0) \neq 0$, then $y_0 \in f(G)$. For each G in O, $f_{0,G}$ lies in F, and if $y_0 \in f_0(G)$, then $d(f_{0,G}, G, y_0) = +1$.*

(b) (*Additivity on domain*) *Suppose that* $f \in F$; f: $\mathrm{cl}(G) \to Y$, *and that* G_1 *and* G_2 *are a pair of disjoint subsets of* O *contained in* G. *Let* $y_0 \notin f(\mathrm{cl}(G) \setminus (G_1 \cup G_2))$. *Then*

$$d(f, G, y_0) = d(f_{G_1}, G_1, y_0) + d(f_{G_2}, G_2, y_0).$$

(c) (*Invariance under homotopy*) *If* $\{f_t\colon 0 \leqslant t \leqslant 1\}$ *is a homotopy in* H *with domain* $\mathrm{cl}(G)$ *for* G *in* O, *and if* $\{y_t\colon 0 \leqslant t \leqslant 1\}$ *is a continuous curve in* Y, *with* $y_t \notin f_t(\partial G)$ *for all* t *in* $[0, 1]$, *then* $d(f_t, G, y_t)$ *is constant in* t *on* $[0, 1]$.

The existence of a degree theory for a given class of mappings is not a trivial assertion. Let us verify this fact through several elementary observations.

PROPOSITION 1. *Let* X *and* Y *be equal to the same Hilbert space* H, *let* O *be a class of open subsets of* H *which includes open balls, and consider a class of mappings* F *and a class of homotopies* H *which includes the affine homotopies in* F (*i.e. homotopies of the form* $f_t = (1-t)f + tf_1$ *for* f *and* f_1 *in* F). *Suppose a degree theory exists for* F *invariant with respect to* H, *and normalized by* $f_0 = I$. *Then*

For any f *in* F *with domain a closed ball* B *about the origin in* H *such that* $(f(u), u) \geqslant 0$ *for all* u *in* B, f *must have a zero in* B.

PROOF OF PROPOSITION 1. Let G be the interior of B. G lies in O and by hypothesis, f lies in the class F. If f has a zero on ∂B, the assertion follows. Otherwise, $d(f, G, 0)$ is well defined. Let f_0 be the restriction of the identity map I to B. Then f_0 lies in F and $d(f_0, G, 0) = +1$. Consider the affine homotopy,

$$f_t = (1-t)f_0 + tf \qquad (0 \leqslant t \leqslant 1).$$

By assumption, this homotopy lies in the class H of permissible homotopies. Hence, if we can show that for all t in $(0, 1)$, $f_t(x) \neq 0$ on ∂B, then $d(f_t, G, 0)$ is constant in t and hence equals $+1$ for all t in $[0, 1]$. Suppose $f_t(x) = 0$ for $0 < t < 1$, with x in ∂B. Then

$$0 = (f_t(x), x) = (1-t)(f_0(x), x) + t(f(x), x).$$

Since $(f(x), x) \geqslant 0$ for all x in ∂G, it follows that

$$0 \geqslant (1-t)(f_0(x), x) = (1-t)\|x\|^2 > 0,$$

which is impossible. Hence $d(f, G, 0) = +1$, and by the normalization property of the degree function, $0 \in f(G)$. Q.E.D.

A corollary of Proposition 1 is the following:

PROPOSITION 2. *Suppose that a degree theory satisfying the hypotheses of Proposition* 1 *exists over a class of mappings* F, *and that for a ball* B *about the origin*, g: $B \to B$ *is a mapping such that* $I - g$ *lies in* F. *Then* g *has a fixed point in* B.

PROOF OF PROPOSITION 2. Since g maps B into B, it follows that for x on ∂B, $\|g(x)\| \leqslant \|x\|$. Hence for $f = I - g$,

$$(f(x), x) = \|x\|^2 - (g(x), x) \geqslant \|x\|^2 - \|g(x)\| \cdot \|x\| \geqslant 0.$$

Applying Proposition 1, f must have a zero in B, i.e. g must have a fixed point in B. Q.E.D.

We draw two conclusions from Proposition 2. First, because we have a degree theory for all continuous mappings on the finite-dimensional space $H = R^n$, all continuous self-mappings of balls in that space have a fixed point, i.e. the Brouwer fixed point theorem. The second conclusion is that for an infinite-dimensional Hilbert space H, we cannot have a degree theory in the sense of Proposition 1 over all continuous mappings since the only case in which all continuous self-maps of the unit ball of a Banach space have a fixed point is that in which the space is finite dimensional.

A variant of Proposition 1 gives us the Poincaré-Miranda theorem stated in the Introduction.

PROPOSITION 3. *Let $X = Y$ be a Banach space, and suppose that for a class of mappings F and a class of homotopies H, we have a degree function defined on F, invariant under H, and normalized by the identity mapping, where H includes all affine homotopies in F. Let G be a set in O, with $0 \in G$, and suppose that f: $\mathrm{cl}(G) \to X$ lies in F. Suppose that for each x in ∂G, there exists a linear functional w_x in X^* such that $\langle w, f(x)\rangle \geq 0$, $\langle w, x\rangle > 0$.*

Then f has a zero in $\mathrm{cl}(G)$.

PROOF OF PROPOSITION 3. Again, we may assume that $0 \notin f(\partial G)$. We construct the homotopy $f_t = (1 - t)I + tf$: $\mathrm{cl}(G) \to X$. By our hypotheses, this homotopy lies in the class H. For $t = 0$, $d(f_0, G, 0) = +1$ and hence to show that $d(f, G, 0) = d(f_1, G, 0) = +1$, it suffices to show that $d(f_t, G, 0)$ is constant in t on $[0, 1]$. This will follow from the property of invariance under homotopy if we can verify that for each t in $(0, 1)$ and all x in $\partial G, f_t(x) \neq 0$.

Suppose, however, that for a given x in ∂G and some t in $(0, 1)$, we have $f_t(x) = 0$. Then

$$0 = \langle w_x, f_t(x)\rangle = (1 - t)\langle w_x, x\rangle + t\langle w_x, f(x)\rangle \geq (1 - t)\langle w_x, x\rangle > 0,$$

which is a contradiction. Q.E.D.

COROLLARY TO PROPOSITION 3. *Let G be the cube $\{x\colon -a_j < x_j < +a_j, j = 1, \ldots, n\}$ in R^n, and let f be a mapping of $\mathrm{cl}(G)$ into R^n such that on the face $C_j^+ = \{x_j = a_j\}$, $f_j(x) \geq 0$, and on the face $C_j^- = \{x_j = -a_j\}$, $f_j(x) \leq 0$. Then f has a zero in* $\mathrm{cl}(G)$.

PROOF OF THE COROLLARY. The boundary of G consists of the union of the faces C_j^+ and C_j^- for various j. We consider R^n as a Hilbert space, and on the faces C_j^+ and C_j^-, we choose w_x to be the element which is 0 except in the jth place, and x_j in the jth place. Then $\langle w_x, f(x)\rangle = x_j f_j(x) \geq 0$, while $\langle w_x, x\rangle = x_j^2 = a_j^2 > 0$. (On the intersections we use whichever of the vectors w_x we please). Hence the hypotheses of Proposition 3 are satisfied, and f has a zero. Q.E.D.

The classical example of an extension of the degree theory for the finite-dimensional case is the Leray-Schauder degree theory which is defined for the case in which $X = Y$ is an arbitrary Banach space, 0 is the class of bounded open subsets of X, F is the class of continuous maps f: $\mathrm{cl}(G) \to X$ with $(I - f)(\mathrm{cl}(G))$ relatively compact in X. Here the class of homotopies $\{f_t: 0 \leqslant t \leqslant 1\}$ is restricted by the assumption that there exists a fixed compact set K in X such that $(I - f_t)(\mathrm{cl}(G)) \subset K$ for all t in $[0, 1]$.

A possible natural extension of the finite-dimensional and the Leray-Schauder degree theories might be a degree theory for proper Fredholm mappings of index zero. Such a theory in the sense of Definition 1 cannot exist.

PROPOSITION 4. *There can be no degree theory in the sense of Definition* 1 *for proper Fredholm mappings of index zero of class* C^∞ *in any infinite-dimensional Hilbert space.*

PROOF OF PROPOSITION 4. The problem, of course, is that all the nonsingular linear mappings of an infinite-dimensional Hilbert space form a single path component (in fact are contractible to a point by a theorem of Kuiper). Suppose that we had a degree theory of the type described. Then given any finite-dimensional subspace R^n of the Hilbert space, we could *suspend* all the differentiable maps on R^n to get Fredholm maps of index zero. The resulting degree theory for maps on R^n obtained by taking the degree of their suspension must be the same as the original degree in R^n, since the uniqueness argument actually works for differentiable mappings of any class. If L_0 is a nonsingular linear mapping of R^n, we would then have

$$d(L, B_n, 0) = d(S(L_0), B, 0) = d(I, B, 0) = +1,$$

since there is a path of nonsingular linear maps leading from any given one to I. This contradicts the fact that for suitably chosen L_0, $d(L_0, B_n, 0) = -1$. Q.E.D.

The proof indicates clearly why the appropriate degree theory over the class of all proper Fredholm operators of index zero must have values in Z_2. (For details of such theories, we refer to the article **[18]**.)

3. The Leray-Schauder theory. The most celebrated form of degree theory for application to nonlinear problems in partial differential equations has been that introduced by Leray and Schauder **[26]** in 1934 for mappings f in infinite-dimensional Banach spaces such that $I - f$ is compact. We shall give a reasonably complete exposition of this theory based upon the results already derived for the finite-dimensional degree.

The basic general result is the following:

THEOREM 3. *Let X be a Banach space, and consider the family F of continuous mappings f: $\mathrm{cl}(G) \to X$, with G a bounded open subset of X, and with $(I - f)(\mathrm{cl}(G))$ relatively compact in X. Let H be the family of continuous homotopies of maps $\{f_t$: $0 \leqslant t \leqslant 1\}$ in F with a common domain* $\mathrm{cl}(G)$ *such that there exists a compact subset K of X with $(I - f_t)(\mathrm{cl}(G)) \subset K$ for all t in* $[0, 1]$. *Then there exists one and only one degree function $d(f, G, y_0)$ in the sense of Definition* 1 *with the identity mapping I as normalizing mapping.*

The uniqueness holds even if the homotopies are restricted to affine homotopies and the mappings to differentiable maps of the form $I - f$ with f having a finite-dimensional range.

The proof of Theorem 3 uses several auxiliary devices which we introduce in the following proposition.

PROPOSITION 5. *Let X be a finite-dimensional Banach space, X_0 a subspace of X, such that X is the direct sum of X_0 and another subspace X_1. Let B_1 be the unit ball about zero in X_1, and for each bounded open subset G_0 of X_0, let $G = G_0 \times B_1$. Let f be a continuous map of* $\mathrm{cl}(G_0)$ *into X_0. Then we define the suspension $S(f)$ mapping* $\mathrm{cl}(G)$ *into X by setting*

$$S(f)(x_0 + x_1) = f(x_0) + x_1, \qquad (x_0 \in \mathrm{cl}(G_0), x_1 \in B_1).$$

Let y_0 be a point of X_0 such that $y_0 \notin f(\partial G_0)$. Then

(1) $y_0 \notin S(f)(\partial G)$.

(2) $d_0(f, G_0, y_0) = d(S(f), G, y_0)$, *where the first degree is calculated in X_0 and the second in X.*

PROOF OF PROPOSITION 5. We set $d_1(f_0, G_0, y_0)$ to be equal to $d(S(f), G, y_0)$. To do this, we first verify (1). Suppose that $x = x_0 + x_1$ has the property that $S(f)(x) = y_0$. This means that

$$f(x_0) + x_1 = y_0$$

or that $x_1 = y_0 - f(x_0)$ lies in X_0. Since x_1 also lies in X_1 and the two subspaces are complementary, $x_1 = 0$ and $f(x_0) = y_0$. Hence x lies in G_0 which does not intersect G. Thus $d(S(f), G, y_0)$ is well defined.

We now examine this new function f_1 on the class of mappings f and note that it satisfies the three conditions for an index function. By Theorem 2, it therefore coincides with the degree function $d(f, G_0, y_0)$. Q.E.D.

PROPOSITION 6. *Let X be a finite-dimensional Banach space, X_0 a subspace of X, G a bounded open set in X such that $G_0 = G \cap X_0$ is nonempty. Let f be a continuous mapping of* $\mathrm{cl}(G)$ *into X, such that $f = I - g$ and* $g(\mathrm{cl}(G)) \subset X_0$. *Let y_0 be a point of X_0 such that $y_0 \notin f(\partial G)$, and let f_0:* $\mathrm{cl}(G_0) \to X_0$ *be the restriction of f to* $\mathrm{cl}(G_0)$.

Then $d(f, G, y_0) = d(f_0, G_0, y_0)$.

PROOF OF PROPOSITION 6. Since ∂G_0 is a subset of ∂G, both degrees are well defined since $y_0 \notin f(\partial G)$.

Let X_1 be a subspace of X complementary to X_0, B_1 the unit ball about 0 in X_1, $G_2 = G_0 \times B_1$. Let $S(f_0)$ be the suspension of f_0 in the sense defined in Proposition 5, with $S(f_0)$ mapping $\mathrm{cl}(G_2)$ into X by the prescription

$$S(f)(x_0 + x_1) = f_0(x_0) + x_1.$$

We shall now consider explicitly for what points x in their respective domains $f(x) = y_0$ and $S(f_0) = y_0$. In the first case, we have

$$x - g(x) = y_0$$

so that if $x = x_0 + x_1$, $x_1 = y_0 - g(x) - x_0 \in X_0$, so that $x_1 = 0$, x lies in G_0, and $f_0(x) = y_0$. Similarly, if $S(f)(x) = y_0$, then

$$x_1 = y_0 - f_0(x_0) \in X_0,$$

$x_1 = 0$, and x lies in G_0 and is a solution of $f_0(x) = 0$. Since $G_0 \subset G \cap G_2$, it follows that

$$d(S(f_0), G_2, y_0) = d(S(f_0), G \cap G_2, y_0); \qquad d(f, G, y_0) = d(f, G \cap G_2, y_0).$$

We now set up the affine homotopy

$$h_t = (1 - t)f + tS(f_0)$$

on $\mathrm{cl}(G \cap G_2)$. If $h_t(x) = 0$ for any t, then we have

$$(1 - t)(x - g(x)) + t(x_1 + f_0(x_0)) = y_0,$$

which may be rewritten as

$$x_1 = y_0 + (1 - t)\{g(x) - x_0\} - tf_0(x_0) \in X_0.$$

Hence $x_1 = 0$, $x = x_0$, $y_0 = f_0(x)$. Thus $x_0 \in G_0$, $x \notin \partial(G_1 \cap G_2)$. Using the invariance of the degree function under homotopy, we see that

$$d(f, G \cap G_2, y_0) = d(S(f_0), G \cap G_2, y_0).$$

Finally, we have

$$\begin{aligned} d(f_0, G_0, y_0) &= d(S(f_0), G_2, y_0) = d(S(f_0), G \cap G_2, y_0) \\ &= d(f, G \cap G_2, y_0) = d(f, G, y_0). \quad \text{Q.E.D.} \end{aligned}$$

LEMMA. *Let K be a compact subset of a Banach space. Then given any $\varepsilon > 0$, there exists a finite-dimensional subset K_ε of X and a continuous mapping p of K into K_ε such that for every x of K, $\|x - p(x)\| < \varepsilon$.*

PROOF OF THE LEMMA. We may form the covering of K by open balls of radius ε. We can find a finite subcovering by compactness and a partition of unity corresponding to this subcovering, i.e., a finite set of points $\{x_1, \ldots, x_r\}$ in K and a family of continuous functions α_j: $K \to [0, 1]$ such that each α_j has its support in the ball of radius ε about x_j and on K, $\Sigma_{j=1}^r \alpha_i = 1$. We set $p(x) = \Sigma_{j=1}^r \alpha_j(x)x_j$. Then $p(x)$ lies in the simplex spanned by the x_j, and for each x, $p(x)$ is a convex linear combination of those x_j lying in the open ball of radius ε about x. Hence $p(x)$ also lies in that ball. Q.E.D.

PROOF OF THEOREM 3. Each f is a proper map of $\mathrm{cl}(G)$ into X. Indeed, suppose K is a compact subset of X, and consider x in $f^{-1}(K)$. Then $x \in g(\mathrm{cl}(G)) + K \subset K_1$, where K_1 is a compact subset of X. Hence $f^{-1}(K)$ is closed in X and a subset of a compact subset of X, and therefore is compact itself. Thus f is proper, and maps closed subsets of $\mathrm{cl}(G)$ into closed subsets of X_0. In particular, $f(\partial G)$ is closed in X and does not contain y_0. Therefore there exists $r > 0$ such that $f(\partial G)$ does not intersect the ball of radius r about y_0.

Using the preceding Lemma, for any $\varepsilon < r$, if we take the compact set K which is the closure of $g(\mathrm{cl}(G))$, we may find a continuous mapping p of K into a finite-dimensional subset of X such that $\|p(u) - u\| < \varepsilon$ for every u in K. We now define a new mapping

$$\tilde{f} = I - \tilde{g}: \mathrm{cl}(G) \to X$$

where $\tilde{g} = p \circ g$. For every x in cl(G), $\|f(x) - \tilde{f}(x)\| < \varepsilon$. In particular, $y_0 \notin \tilde{f}(\partial G)$.

Take any finite-dimensional subspace X_0 of X which contains g_0 and $\tilde{g}(\mathrm{cl}(G))$, and let $G_0 = G \cap X_0$. Then by Proposition 6, $d(f_{G_0}, G_0, y_0)$ is independent of the choice of the subspace. We propose to show that it is also independent of the choice of the approximation.

Take two such approximations $f_1 = I - g_1$ and $f_2 = I - g_2$ with $\|g_1(x) - g(x)\| < r$, $\|g_2(x) - g(x)\| < r$ for every x in cl(G) and with the values of g_1 and g_2 lying in finite-dimensional subspaces of X. If we set $g_t = (1 - t)g_1 + tg_2$, then each g_t is another such approximation with the values of all g_t lying in a fixed finite-dimensional subspace X_0. If we let X_0 also contain y_0, then $I - g_t$ restricted to cl(G_0) with $G_0 = G \cap X_0$, is a permissible homotopy, and $d(I - g_t, G_0, y_0)$ is independent of t in $[0, 1]$. It follows that $d(f_1, G_0, y_0) = d(f_2, G, y_0)$.

We define the degree function $d(f, G, y_0)$ as the common value of $d(f, G_0, y_0)$ for any of the approximating mappings and the corresponding finite-dimensional subspaces X_0 containing y_0 and $g(\mathrm{cl}(G))$. The normalization condition and the additivity on domain follow easily by choosing the approximating mappings sufficiently close to f. For a permissible homotopy $f_t = I - g_t$, with $g_t(\mathrm{cl}(G)) \subset K$ for a fixed compact set K and all t in $[0, 1]$, we choose $r > 0$ such that the ball of radius r about y_t does not meet $f_t(\partial G)$ for any t in $[0, 1]$ and then apply the approximation operator p for K with $\varepsilon < r$ to all the g_t. Then $\{\tilde{g}_t: 0 \leqslant t \leqslant 1\}$ is a continuous homotopy of mappings with values in a finite-dimensional subspace X_0 of X. If $G_0 = G \cap X_0$, and we have absorbed y_t into the g_t, we see that $d(\tilde{f}_t, G_0, 0)$ is independent of t in $[0, 1]$. Thus all the conditions for the degree function have been satisfied.

Suppose now that we have two degree functions d and d_1. By the finite-dimensional argument, d and d_1 must give the same values for suspensions of mappings of finite-dimensional subspace of X. Hence by the proof of Proposition 6, the two degree functions give the same result for mappings $f = I - g$, with $g(\mathrm{cl}(G))$ lying in a finite-dimensional subspace of X. Since every f of the form $I - g$ can be uniformly approximated by mappings of this last type and is affinely homotopic to such nearby mappings with homotopy paths on the boundary that avoid y_0, it follows that the two degree functions must coincide. Thus the proof of Theorem 3 is complete. Q.E.D.

4. The direct method of the calculus of variations and mappings of monotone type. Let X be a Banach space, G an open subset of X. If φ is a real-valued differentiable function on G, then its derivative at any point x_0 of G is given by

$$\lim_{\varepsilon \to 0} \frac{\varphi(x_0 + \varepsilon v) - \varphi(x_0)}{\varepsilon} = \langle \varphi'(x_0), v \rangle$$

which depends linearly and continuously upon the direction v in X and hence is an element of the conjugate space X^* of X.

We use here some of the standard notation of this kind of functional analysis. The duality between the real Banach spaces X^* and X is given by $\langle w, x\rangle$ for w in X^* and x in X. We shall use the symbol $\to$ for strong convergence, $\rightharpoonup$ for weak convergence.

The derivative φ' thus gives us a mapping from G, a subset of X, to X^*, the conjugate space of X. Thus mappings from subsets of X to X^* are a rather natural framework for any approach from functional analysis which attempts to encompass the direct method of the calculus of variations. The latter term, which goes back to the Dirichlet principle of Riemann and its rigorization by Hilbert in 1900, refers to the process of trying to get solutions of the equation $\varphi'(x) = 0$ by finding local maxima or minima of the functional $\varphi(x)$. The basic assumption which is necessary for the case when the problem is imposed in an infinite-dimensional context goes back in its origins to Hilbert and involves using a convexity assumption (or some form of modified convexity) upon the functional φ. The argument appears almost truistic in its modern form because it has been essentially absorbed into the structure of our basic functional analysis.

DEFINITION 2. *Let X be a Banach space, G a subset of X, f a mapping of G into X^*. Then*

(a) *f is said to be monotone if for all u and v of G,*

$$\langle f(u) - f(v), u - v\rangle \geq 0.$$

(b) *f is said to be of class $(S)_+$ if for any sequence $\{x_j\}$ in G which converges weakly to x in X and for which $\overline{\lim}\langle f(x_j), x_j - x\rangle \leq 0$, we have $x_j \to x$.*

(c) *f is said to be pseudo-monotone, if for any sequence $\{x_j\}$ in G for which $x_j \rightharpoonup x$ for some x in X while $\overline{\lim}\langle f(x_j), x_j - x\rangle \leq 0$, we have $\lim\langle f(x_j), x_j - x\rangle = 0$, and if $x \in G$, then $f(x_j) \rightharpoonup f(x)$.*

PROPOSITION 7. (a) *If G is a convex open subset of X and φ is a C^1 real-valued function on G, then φ is convex on G if and only if $f = \varphi'$ is a monotone mapping of G into X.*

(b) *Suppose that φ is a C^1 functional on G, $f = \varphi'$. If f is bounded and pseudo-monotone, then φ is weakly sequentially lower semicontinuous on G, i.e. if $\{x_j\}$ is a sequence in G which converges weakly to x in G, then $\varphi(x) \leq \underline{\lim}\, \varphi(x_j)$.*

PROOF OF PROPOSITION 7. *Proof of* (a). Suppose first that φ' is monotone. For u and v two points of G and s in [0, 1], set

$$q(s) = \varphi(su + (1 - s)v) - s\varphi(u) - (1 - s)\varphi(v).$$

To show that φ is convex, we must show that for every u, v and s, $q(s) \leq 0$. We see that $q(0) = 0 = q(1)$. On the other hand, q is continuously differentiable and

$$q'(s) = \langle \varphi'(v + s(u - v)), u - v\rangle + \varphi(v) - \varphi(u).$$

Hence for $0 \leqslant s < t \leqslant 1$, we have

$$q'(t) - q'(s) = \langle \varphi'(v_t) - \varphi(v_s), u - v \rangle$$

where $v_t = v + t(u - v)$, $v_s = v + s(u - v)$. Since $v_t - v_s = (t - s)(u - v)$, it follows from monotonicity that

$$q'(t) - q'(s) = (t - s)^{-1} \langle \varphi'(v_t) - \varphi'(v_s), v_t - v_s \rangle \geqslant 0.$$

Suppose that $\max q(s) > 0$. Then q has an interior maximum at s_0 with $q'(s_0) = 0$. Since $q(s_0) > 0$, and $q'(t) \geqslant 0$ for all $t > s_0$, it is impossible that $q(1)$ could be zero. Hence $q(s) \leqslant 0$ for all s in $[0, 1]$ and q is a convex function.

Suppose conversely that φ is convex. Then for u and v in G and $0 < s < 1$, we would have

$$\frac{\varphi(v + s(u - v)) - \varphi(v)}{s} \leqslant \varphi(u) - \varphi(v).$$

Letting $s \to 0+$, we obtain

$$\langle \varphi'(v), u - v \rangle \leqslant \varphi(u) - \varphi(v).$$

Interchanging u and v, we also have

$$\langle \varphi'(u), v - u \rangle \leqslant \varphi(v) - \varphi(u).$$

Adding we find that

$$\langle \varphi'(u) - \varphi(v), u - v \rangle \geqslant 0,$$

i.e. φ' is monotone.

Proof of (b). For each t in $[0, 1]$, set $v_{j,t} = x + t(x_j - x)$. Then $v_{j,t} \rightharpoonup x$ as $j \to +\infty$ for each fixed t, while $v_{j,t} - x = t(x_j - x)$. For $t > 0$, we have therefore by the pseudo-monotonicity of φ',

$$\underline{\lim} \langle \varphi'(v_{j,t}), x_j - x \rangle = \underline{\lim} \langle \varphi'(v_{j,t}), t^{-1}(v_{j,t} - x) \rangle \geqslant 0.$$

For each j, we have

$$\varphi(x_j) - \varphi(x) = \int_0^1 \langle \varphi'(v_{j,t}), x_j - x \rangle \, dt \geqslant \int_0^1 \{ \langle \varphi'(v_{j,t}), x_j - x \rangle \}^- \, dt,$$

where the integrand in the last integral is uniformly bounded and converges to 0 as $j \to +\infty$ for each $t > 0$. Hence $\underline{\lim}\, \varphi(x_j) - \varphi(x) \geqslant 0$. Q.E.D.

The primary interest of the definitions for the class $(S)_+$ and for pseudo-monotonicity arises from the fact that one can verify these properties under suitable concrete hypotheses for the maps of a Sobolev space $W_0^{m,p}(\Omega)$ into its conjugate space $W^{-m,p'}(\Omega)$ obtained from an elliptic operator in generalized divergence form

$$A(u) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \ldots, D^m u).$$

Here Ω is an open subset of a Euclidean space R^n, $n \geqslant 1$, and p is an exponent in the reflexive range $1 < p < +\infty$. The Sobolev space $W^{m,p}(\Omega)$ is the space of u in $L^p(\Omega)$ (with respect to Lebesgue n-measure) with all the derivatives $D^\alpha u$ also in

$L^p(\Omega)$ for all distribution derivatives of order $\leq m$. The norm of this space is obtained by injecting it into the product of L^p-spaces, one for each derivative, by the jet-mapping

$$u \to \{D^\alpha u: |\alpha| \leq m\}.$$

The closed subspace $W_0^{m,p}(\Omega)$ is obtained by taking the closure in $W^{m,p}(\Omega)$ of the testing functions with compact support in Ω.

The operator A given by the differential expression written above makes sense as a mapping of $W_0^{m,p}(\Omega)$ into its conjugate space $W^{-m,p'}(\Omega)$, with $p' = p/(p-1)$ the conjugate exponent to p, provided that if we replace u and its derivatives by the algebraic variable

$$\xi = \{\xi_\alpha: |\alpha| \leq m\},$$

then $A_\alpha(x, \xi)$ is measurable in x for fixed ξ and continuous in ξ for fixed x in Ω, and satisfies an inequality of the form

$$|A_\alpha(x, \xi)| \leq c(|\xi|^{p-1} + k_0(x))$$

with k_0 a function in $L^{p'}(\Omega)$. Under these conditions, if we substitute for u and its derivatives in the formal expression, functions in $L^p(\Omega)$, $A_\alpha(x, u, \ldots, D^m u)$ yields an element of $L^{p'}(\Omega)$. Its distribution derivative $D^\alpha A_\alpha(x, \xi(u))$ then becomes an element of the space of distributions $W^{-m,p'}(\Omega)$.

Further hypotheses of some sort of ellipticity and of weak semiboundedness must be imposed before this operator A from $W = W_0^{m,p}(\Omega)$ to X is either pseudo-monotone or of class $(S)_+$. Such conditions for example are

$$\sum_{|\alpha|=m} A_\alpha(x, \eta, \zeta) - A_\alpha(x, \eta, \zeta^\#)(\zeta_\alpha - \zeta_\alpha^\#) > 0$$

for $\zeta \neq \zeta^\#$, where ξ is broken up into the mth order piece ζ and the lower order piece η.

$$\sum_{|\alpha|\leq m} A_\alpha(x, \xi)\xi_\alpha \geq c_0|\xi|^p - k_1(x) \qquad (k_1 \in L^1(\Omega)).$$

Under these hypotheses, A is of class $(S)_+$. Under very much weaker conditions replacing the semiboundedness in the second hypothesis, A remains pseudo-monotone.

If $X = X^*$ is a Hilbert space H, the mappings of type $(S)_+$ contain as a special case the Leray-Schauder maps $I - g$, with g compact. This follows from the fact that each strongly monotone mapping f is of type $(S)_+$ and the fact that the class $(S)_+$ is always invariant under compact perturbations. We say that f is strongly monotone if there exists a continuous positive increasing function $p(r)$ for $r > 0$ such that

$$\langle f(u) - f(v), u - v\rangle \geq p(\|u - v\|).$$

For the detailed development of the discussion of later sections, we need to develop the properties of our *canonical* monotone mapping, the duality mapping J, which under suitable hypotheses is also a map of class $(S)_+$. In our further discussion, we shall always assume that the Banach space X with which we are dealing is reflexive. This is certainly true for the Sobolev spaces $X = W_0^{m,p}(\Omega)$ which in fact are uniformly convex with duals which are also uniformly convex. It is not always true that every reflexive Banach space has an equivalent norm which is uniformly convex, and a fortiori that X is also uniformly convex. By results due to Lindenstrauss, Asplund, and Trojanski however, it is true that X can be renormed so that X and X^* are both locally uniformly convex. We shall use only one consequence of this renorming: In the resulting norms on both X and X^*, if a sequence $\{x_j\}$ converges weakly to x in X and $\|x_j\| \to \|x\|$, then x_j converges strongly to x, with a similar property for sequences $\{w_j\}$ in X^*.

PROPOSITION 8. *Let X be a reflexive Banach space which is normed so that both X and X^* are locally uniformly convex (and in particular are strictly convex). Then there exists a unique bicontinuous mapping J of X onto X^* which is given by the conditions that for each x of X, $\|J(x)\| = \|x\|$ and $\langle J(x), x\rangle = \|x\|^2$. This mapping J, called the duality mapping corresponding to the given norm on X, is both monotone and a mapping of class $(S)_+$.*

PROOF OF PROPOSITION 8. For each x in X, it follows from the Hahn-Banach theorem that there exists w in X such that $\|w\| = \|x\|$ and $\langle w, x\rangle = \|w\| \cdot \|x\|$. Such elements w lie on the sphere $\|w\| = \|x\|$ and are also characterized by the inequalities

$$\langle w, x\rangle = \|x\|^2, \qquad \|w\| \leqslant \|x\|.$$

Hence they form a convex subset of a sphere in X^*. If X^* is strictly convex, such an element w is unique, and this unique element is $J(x)$ for the given x.

For each x and u in X, we may compute

$$\begin{aligned}\langle J(x) - J(u), x - u\rangle &= \|x\|^2 + \|u\|^2 - \langle J(x), u\rangle - \langle J(u), x\rangle \\ &= (\|x\| - \|u\|)^2 + \{\|u\| \cdot \|J(x)\| - \langle J(x), u\rangle\} \\ &\quad + \{\|J(u)\| \cdot \|x\| - \langle J(u), x\rangle\}.\end{aligned}$$

All the parentheses are nonnegative. Hence J is monotone. Moreover, if we replace u by x_j, we obtain

$$\begin{aligned}\langle J(x_j) - J(x), x_j - x\rangle &\geqslant (\|x_j\| - \|x\|)^2 \\ &\geqslant \left|\langle J(x_j), x\rangle - \|J(x_j)\| \cdot \|x\|\right| \\ &\geqslant \left|\langle J(x), x_j\rangle - \|J(x)\| \cdot \|x_j\|\right|.\end{aligned}$$

Suppose $x_j \rightharpoonup x$. Then $\langle J(x), x_j - x\rangle \to 0$. Suppose moreover that

$$\overline{\lim} \langle J(x_j), x_j - x\rangle \leqslant 0.$$

Then $\|x_j\| \to x$, and it follows that $x_j \to x$. Hence J lies in the class $(S)_+$.

Suppose that $x_j \to x$. Then $(J(x_j) - J(x), x_j - x) \to 0$. Hence $\langle J(x_j), x\rangle \to \|x\|^2$. Choose any weakly convergent subsequence $J(x_j) \rightharpoonup w$. Then $\langle w, x\rangle = \|x\|^2$

and $\|w\| \leq x$. Hence $w = J(x)$. Moreover $\|w\| = \lim\|J(x_j)\|$ and therefore $J(x_j) \to w = J(x)$. Thus J is continuous from X to X^*. If on the other hand, $J(x_j) \to J(x)$, it also follows that $\langle J(x_j) - J(x), x_j - x\rangle \to 0$, so that $\|x_j\| \to \|x\|$, while

$$\langle J(x), x_j \rangle \to \|x\|^2.$$

It follows by the same argument that $x_j \to x$. Hence J^{-1} is continuous from X^* to X. Q.E.D.

5. The degree of mapping for mappings of class $(S)_+$. In the present section, we devote our efforts to the proof of the following theorem.

THEOREM 4. *Let X be a reflexive Banach space, and consider the family F of maps f: $\mathrm{cl}(G) \to X^*$, where G is a bounded open subset of X and f is a mapping of class $(S)_+$ with f demicontinuous (i.e. continuous from the strong topology of X to the weak topology of X^*). Let H be the class of affine homotopies in F, and let J be the duality mapping from X to X^* corresponding to an equivalent norm on X in which both X and X^* are locally uniformly convex.*

Then there exists one and only one degree function on F which is invariant under H and normalized by the map J.

The homotopies described in Theorem 4 are too weak for some important applications of this result. We therefore introduce the following broader class.

DEFINITION 3. *Let G be a bounded open subset of X, $\{f_t\colon 0 \leq t \leq 1\}$ a one-parameter family of maps of $\mathrm{cl}(G)$ into X^*. Then $\{f_t\}$ is said to be a homotopy of class $(S)_+$ if it satisfies the following condition: For any sequence $\{x_j\}$ in $\mathrm{cl}(G)$ converging weakly to some x in X and for any sequence $\{t_j\}$ in $[0, 1]$ converging to t for which*

$$\overline{\lim} \left\langle f_{t_j}(x_j), x_j - x \right\rangle \leq 0,$$

we have x_j converging strongly to x and $f_{t_j}(x_j) \rightharpoonup f_t(x)$.

THEOREM 5. *The degree function described in Theorem 4 is invariant under homotopies of class $(S)_+$.*

The construction of the degree function for maps of class $(S)_+$ and the proofs of Theorems 4 and 5 rest upon the general technique of Galerkin approximation.

DEFINITION 4. *Let X be a Banach space, X_0 a closed subspace of X, G an open subset of X such that $G \cap X_0 = G_0$ is nonempty. Let f: $\mathrm{cl}(G) \to X$ be a given mapping. Then the Galerkin approximant for the map f is the mapping f_0: $\mathrm{cl}(G_0) \to X_0$, where φ is the injection mapping of X_0 into X, φ^* the corresponding projection of X^* onto X_0^*, and f_0 is given by*

$$f_0(x) = \varphi^*(f(x)) \qquad (x \in \mathrm{cl}(G_0)).$$

In a sort of inverse relationship to the concept of the Galerkin approximant is the following notion of a generalized *suspension* operator.

DEFINITION 5. *Let X be a reflexive Banach space, J a duality mapping of X into X^* of class $(S)_+$. Suppose that X_0 is a closed subspace of X having a closed complement X_1 in X, P the projection of X on X_0 which corresponds to this splitting so that $(I - P)$ is the projection on X_1. We consider P as an element of the space of bounded linear maps of X, with P_1 the same projection considered as a bounded linear map of X on X_0. Then P^* is a bounded linear map of X^* into itself, P_1^* the injection of X_0^* into X^* whose image is the same as the range of the projection P^*. Let B_1 be the unit ball about 0 in X_1.*

Then

(1) *For each bounded open subset G_0 of X_0, we define its suspension $S(G_0)$ as the bounded open subset of X given by*

$$S(G_0) = P^{-1}(G_0) \cap (I - P)^{-1}(B_1).$$

Thus each element x of $S(G_0)$ can be written uniquely in the form

$$x = x_0 + x_1 \qquad (x_0 \in G_0, x_1 \in B_1)$$

where $x_0 = Px$, $x_1 = (I - P)(x)$.

(2) *If g: $\mathrm{cl}(G_0) \to X_0^*$ is given, we define its suspension S_g as the map of $\mathrm{cl}(S(G_0)) \to X^*$ given by*

$$S_g(x) = P_1^* g P(x) + (I - P)^* J(I - P)(x).$$

PROPOSITION 9. *Under the circumstances described in Definition* 5:

(a) *If g is demicontinuous and of class $(S)_+$ from $\mathrm{cl}(G_0)$ to X_0^*, then S_g is demicontinuous and of class $(S)_+$ from $\mathrm{cl}(S(G_0))$ to X^*.*

(b) *If $\{g_t\}$ is an affine homotopy on $\mathrm{cl}(G_0)$, then $\{S_{g_t}\}$ is an affine homotopy on $\mathrm{cl}(S(G_0))$.*

(c) *If the point y_0 in X_0^* does not lie in $g(\partial G_0)$, then $P_1^*(y_0)$ does not lie in $S_g(\partial S(G_0))$.*

PROOF OF PROPOSITION 9. *Proof of* (a). Let $\{u_j\}$ be a sequence in $\mathrm{cl}(S(G_0))$ such that $u_j \rightharpoonup u$ while $\overline{\lim}\langle S_g(u_j), u_j - u\rangle \leq 0$. Let $v_j = Pu_j$, $x_j = (I - P)u_j$. Then $v_j \rightharpoonup v = Pu$, $x_j \rightharpoonup x = (I - P)u$, with v in X_0, x in X_1. Using the definition of S_g, we see that

$$\langle S_g(u_j), u_j - u\rangle = \langle g(v_j), v_j - v\rangle + \langle J(x_j), x_j - x\rangle.$$

Since $\underline{\lim}\langle J(x_j), x_j - x\rangle \geq 0$, it follows that $\overline{\lim}\langle g(v_j), v_j - v\rangle \leq 0$. Hence $v_j \to v$ and $g(v_j) \rightharpoonup g(v)$, implying that $\langle g(v_j), v_j - v\rangle \to 0$. Hence

$$\overline{\lim}\,\langle J(x_j), x_j - x\rangle \leq 0.$$

Since J is of class $(S)_+$, $x_j \to x$. Therefore $u_j = x_j + v_j \to x + v = u$. Q.E.D.

Proof of (b). This follows immediately from the linearity of P and P and the definition of the suspension of a map.

Proof of (c). Suppose for x in $\mathrm{cl}(S(G_0))$ that $S_g(x) = P_1^*(y_0)$. We remark that $P^*(X^*) = P_1^*(X_0^*)$ is a closed complement to $(I - P)^*(X^*)$. Thus if

$$P_1^*(g(Px) - y_0) + (I - P)^* J(I - P)(x) = 0,$$

it follows that

$$P_1^*(g(Px)-y_0)=0, \qquad (I-P)^*J(I-P)(x)=0.$$

Hence, from the second equation, we find that

$$0=\langle (I-P)^*J(I-P)(x),x\rangle=\langle J(I-P)x,(I-P)x\rangle=\|(I-P)x\|^2,$$

so that $x=Px$. From the other equation, we see that since P_1^* is injective, $g(x)-y_0=g(Px)-y_0=0$. By the assumption that $y_0 \notin g(\partial G_0)$, and since $x=Px$ lies in $\mathrm{cl}(G_0)$, we see that x lies in G_0. In particular, $P_1^*(y_0)$ does not lie in $S_g(\partial S(G_0))$ since $G_0 \subset S_g(G_0)$. Q.E.D.

We can use Proposition 9 in the following way to obtain an important methodological conclusion about degree functions for maps of class $(S)_+$.

Proposition 10. *Under the circumstances of Definition 5 and Proposition 9, suppose that for both of the spaces X and X_0, we have degree functions d and d_0 defined on the demicontinuous mappings of class $(S)_+$ and invariant with respect to affine homotopies in each case, with the degree functions normalized by J and J_0, respectively, where J and J_0 are the appropriate duality maps. Suppose moreover that the degree function d_0 is uniquely characterized on the maps of class $(S)_+$ by these properties. Then for each map g: $\mathrm{cl}(G_0)\to X_0^*$ demicontinuous and of class $(S)_+$ and for each y_0 in X_0^* such that $y_0 \notin g(\partial G_0)$, we have*

$$d_0(g,G_0,y_0)=d\big(S_g,S(G_0),P_1(y_0)\big).$$

Proof of Proposition 10. For each g and y_0 as above, we define a new degree function using Proposition 9 by setting

$$d_1(g,G_0,y_0)=d\big(S_g,S(G_0),P_1^*(y_0)\big).$$

This gives a degree function which is invariant under affine homotopies on the demicontinuous maps g of class $(S)_+$. We remark that the normalization condition is always equivalent to

$$d_1(J_0,B_0,0)=+1$$

for the unit ball B_0 with center 0 in X_0. The map $f=S_{J_0}$ satisfies the condition $\langle f(u),u\rangle>\theta$ for $u\neq 0$. If $G=S(B_0)$, $0\in G$ and the affine homotopy $\{f_t=(1-t)f+tJ\}$ has no zeroes except at $u=0$. Hence

$$d\big(S_{J_0},S(B_0),0\big)=d(J,S(B_0),0)=+1.$$

Since d_0 is the unique degree function for maps of class $(S)_+$ on X_0, it follows that $d_0=d_1$. Q.E.D.

We apply this result to the case in which X_0 is a finite-dimensional Banach space. Then X_0 is equivalent to a Hilbert space and X_0^* can be identified with X_0. By the remark we just made, normalization by the identity mapping on a finite-dimensional space is equivalent to normalization by any mapping f which is bicontinuous and such that $\langle f(u),u\rangle>0$ for $u\neq 0$, and in particular by the duality map corresponding to the original norm on X_0. Hence

COROLLARY TO PROPOSITION 10. *Let X be a reflexive Banach space, X_0 a finite-dimensional subspace of X. Suppose that there exists a degree function d on the demicontinuous maps of class $(S)_+$ in the space X. Then for each continuous map g: $\mathrm{cl}(G_0) \to X_0^*$ where G_0 is a bounded open subset of X_0 and for each y_0 outside $g(\partial G_0)$, we have S_g demicontinuous and of class $(S)_+$ mapping $\mathrm{cl}(S(G_0))$ into X^* and*

$$d_0(g, G_0, y_0) = d\big(S_g, S(G_0), P_1^*(y_0)\big).$$

PROOF OF THE COROLLARY. Since X_0 is of finite dimension, it has a closed complement X_1 in X, so that the suspension operator of Definition 5 is well defined for some projection map P of X on X_0. Moreover, for finite-dimensional spaces X_0, each continuous map g is demicontinuous and of class $(S)_+$ (indeed the two classes coincide). Moreover, as we already observed, we know the uniqueness of the degree function by the results of §1. Hence the desired conclusion follows from Proposition 10. Q.E.D.

We use the Corollary to Proposition 10 as an essential tool in the proof of our basic auxiliary result, Proposition 11.

PROPOSITION 11. *Let X be a reflexive Banach space, X_0 a finite-dimensional subspace of X, and suppose that a degree function d is given on the demicontinuous maps of class $(S)_+$ defined on the closures of bounded open sets G in X with values in X^*. Suppose that this degree is invariant under affine homotopies and is normalized by a duality map J of class $(S)_+$.*

Let f: $\mathrm{cl}(G) \to X^*$ be a map in the given class, with $0 \in G$ and let f_0 be the Galerkin approximant of f with respect to X_0, so that f_0: $\mathrm{cl}(G_0) \to X_0^*$, with $G_0 = G \cap X_0$. Suppose either that $d(f, G, 0)$ is not defined, or $d_0(f_0, G_0, 0)$ is not defined, or (if both are well defined), then

$$d(f, G, 0) \neq d_0(f_0, G_0, 0)$$

where we use d_0 to denote the finite-dimensional degree in X_0.

Then: There exists u in ∂G such that

$$\langle f(u), u\rangle \leqslant 0,$$

and for all v in X_0,

$$\langle f(u), v\rangle = 0.$$

PROOF OF PROPOSITION 11. Suppose first that for some u_0 in ∂G, $f(u_0) = 0$. Then u_0 satisfies our conclusion. Suppose next that for some u_0 in ∂G_0, $f_0(u_0) = 0$. Then again u_0 satisfies the conclusion. Thus we may assume without loss of generality that both $d(f, G, 0)$ and $d(f_0, G_0, 0)$ are well defined (i.e. $0 \notin f(\partial G)$, $0 \notin f_0(\partial G_0)$), and

$$d(f, G, 0) \neq d_0(f_0, G_0, 0).$$

If we apply the Corollary of Proposition 10, we see that

$$d_0(f_0, G_0, 0) = d\big(S_{f_0}, S(G_0), 0\big).$$

We define an auxiliary mapping f_1: cl$(G) \to X^*$ by setting

$$f_1(x) = P^*f(x) + (I - P)^*J(I - P)(x).$$

We have already noted in the proof of Proposition 9 that since 0 lies in X_0, all the solutions x of the equation $S_{f_0}(x) = 0$ must lie in G_0 and hence in $G \cap S(G_0)$. Using the additivity property of the degree, we see that

$$d\big(S_{f_0}, S(G_0), 0\big) = d\big(S_{f_0}, S(G_0) \cap G, 0\big).$$

Similarly, suppose for x in cl(G), we have $f_1(x) = 0$. Then we have

$$(I - P)^*J(I - P)(x) = 0,$$

from which it follows that $(I - P)x = 0$, x lies in cl(G_0), and $P^*f(x) = 0$. For each v in X_0, it follows that

$$\langle f(x), v \rangle = \langle P^*f(x), v \rangle = 0,$$

so that $f_0(x) = 0$. Thus x must lie in G_0, since by our assumption $0 \notin f_0(\partial G_0)$.

We note next that f_1 is a mapping of class $(S)_+$ from cl(G) to X. The first summand is compact since P is of finite dimension, so that it suffices to show that $(I - P)^*J(I - P)$ is of class $(S)_+$. Suppose that $u_j \rightharpoonup u$, and that

$$\overline{\lim} \big\langle (I - P)J(I - P)u_j, u_j - u \big\rangle \leqslant 0.$$

If $x_j = (I - P)u_j$, then $x_j \rightharpoonup x = (I - P)u$ by the fact that $(I - P)$ is a continuous linear map. Moreover

$$\big\langle (I - P)^*J(I - P)u_j, u_j - u \big\rangle = \langle Jx_j, x_j - x \rangle.$$

Thus

$$\overline{\lim} \langle Jx_j, x_j - x \rangle \leqslant 0.$$

Since J is of class $(S)_+$, it follows that $x_j \to x$. Since P is compact, we know also that $Pu_j \to Pu$. Hence $u_j = x_j + Pu_j \to x + Pu = u$, and we have shown that f_1 is of class $(S)_+$. Since f_1 is obviously also continuous, the degree function d applies to the mapping f_1. Hence

$$d(f_1, G, y_0) = d(f_1, G \cap G_2, y_0).$$

We now consider the affine homotopy $h_t = (1 - t)f_1 + tS_{f_0}$ on $G \cap G_2$. For any t in $(0, 1)$, suppose that for some x, $h_t(x) = 0$. This means that

$$(1 - t)P^*f(x) + tP_1^*f_0(Px) + (I - P)^*J(I - P)x = 0.$$

Again this means that both summands must equal 0, i.e. in particular that $(I - P)^*J(I - P)x = 0$. This means once more that $x = Px$. Thus for every v in X_0,

$$0 = \big\langle (1 - t)P^*f(x) + tP^*f_0(x), v \big\rangle = \langle f_0(x), v \rangle.$$

Thus x lies in cl(G_0) and $f_0(x) = 0$, so that by our initial assumption x lies in G_0. In particular, h_t has no zeroes for any t on the boundary of $G \cap G_2$. Hence

$$d(f_1, G \cap G_2, 0) = d\big(S_{f_0}, G \cap G_2, 0\big).$$

Combining the various equalities we have proved, we obtain

$$d(f_0, G_0, 0) = d(S_{f_0}, G_2, 0) = d(S_{f_0}, G \cap G_2, 0)$$
$$= d(f_1, G \cap G_2, 0) = d(f_1, G, 0).$$

If we combine this with our initial assumption that $d(f_0, G_0, 0) \neq d(f, G, 0)$, we find that

$$d(f, G, 0) \neq d(f_1, G, 0).$$

Now form the affine homotopy $k_t = (1-t)f + tf_1$. By the inequality of the degrees on G for f and f_1 over 0, we see that there must exist u_0 in ∂G and t in $(0, 1)$ such that $k_t(u_0) = 0$.

For any v in X_0, we see that

$$0 = \langle k_t(u_0), v \rangle = (1-t)\langle f(u_0), v \rangle + t\langle f(u_0), v \rangle = \langle f(u_0), v \rangle.$$

Hence, we obtain the desired equality

$$\langle f(u_0), v \rangle = 0 \qquad (v \in X_0).$$

Now consider the equality

$$\begin{aligned} 0 &= \langle k_t(u_0), (I-P)u_0 \rangle \\ &= (1-t)\langle f(u_0), (I-P)u_0 \rangle + t\langle J(I-P)u_0, (I-P)u_0 \rangle \\ &= (1-t)\langle f(u_0), u_0 \rangle + t\|(I-P)u_0\|^2. \end{aligned}$$

Hence

$$\langle f(u_0), u_0 \rangle = -t(1-t)^{-1}\|(I-P)u_0\|^2 \leq 0. \quad \text{Q.E.D.}$$

PROOF OF THEOREM 4. We carry through the proof both for the existence and for the uniqueness of the degree function for maps of class $(S)_+$ by applying the Galerkin approximation method over the partially ordered set of finite-dimensional subspaces of X.

Let Λ be this partially ordered set of finite-dimensional subspaces X_λ ordered by inclusion. For each λ, let φ_λ be the injection map of X_λ into X and φ_λ^* the corresponding projection map of X^* onto X_λ^*. We consider only finite-dimensional spaces such that $G_\lambda = G \cap X_\lambda$ is nonempty. We assume that $y_0 = 0$, which does not affect the generality of the argument since we can replace each map f by $f - y_0$. We now assert

There exists λ_0 in Λ such that for all $\lambda > \lambda_0$, $0 \notin f_\lambda(\partial G_\lambda)$ and $d(f_\lambda, G_\lambda, 0)$ is independent of λ.

Suppose otherwise. Then for each X_λ, there would exist $X_\mu \supset X_\lambda$ such that

$$d(f_\mu, G_\mu, 0) \neq d(f_\lambda, G_\lambda, 0)$$

(or one of the two degrees is not defined). In all these cases, we may apply Proposition 11 with X replaced by X_μ and X_λ considered as a subspace of X_μ. Then f_λ is obviously the Galerkin approximant of f_μ with respect to X_λ. Since both X_μ and X_λ have degree functions defined, and the uniqueness of degree holds for

X_λ, we apply Proposition 11 and obtain a point u in ∂G_μ such that

$$\langle f(u), u \rangle = \langle f_\mu(u), u \rangle \leq 0,$$

while for all v in X_λ,

$$\langle f(u), v \rangle = \langle f_\mu(u), v \rangle = 0.$$

Let us now define a subset V_λ of ∂G by

$$V = \{u \mid u \in \partial G, \langle f(u), u \rangle \leq 0; \langle f(u), v \rangle = 0 \text{ for all } v \text{ in } X_\lambda\}.$$

By our preceding paragraph, each V_λ is nonempty. The family $\{V_\lambda\}$ is contained in a fixed bounded set ∂G and it has the finite intersection property. If we denote by w-cl(V_λ) the closure of V in the weak topology of X, then w-cl(V_λ) is weakly compact, and

$$\bigcap_\lambda \text{w-cl}(V_\lambda) \neq \varnothing.$$

Let x_0 be an arbitrary point in this intersection.

Let v be an arbitrary point in X, and choose a subspace X_λ in the ordered set Λ such that X_λ contains both x_0 and v. Since x_0 lies in w-cl(V_λ) and the space X is reflexive, there exists a sequence $\{u_j\}$ in V_λ which converges weakly to x_0 in X. Such u_j must lie in ∂G by the definition of V_λ. Moreover, for each j,

$$\langle f(u_j), u_j \rangle \leq 0, \qquad \langle f(u_j), x_0 \rangle = 0, \qquad \langle f(u_j), v \rangle = 0.$$

In particular,

$$\langle f(u_j), u_j - x_0 \rangle \leq 0,$$

and a fortiori, $\overline{\lim}\langle f(u_j), u_j - x_0 \rangle \leq 0$. Using the assumption that f lies in the class $(S)_+$, we see that u_j converges strongly to x_0 and hence x_0 lies in ∂G. By the demicontinuity of f, $f(u_j) \rightharpoonup f(x_0)$. Hence

$$\langle f(x_0), v \rangle = \lim \langle f(u_j), v \rangle = 0.$$

Since v was an arbitrary element of X, it follows that $f(x_0) = 0$. Thus we reach a contradiction with the assumption that $0 \notin f(\partial G)$.

We define $d(f, G, 0)$ as the common value of $d(f_\lambda, G_\lambda, 0)$ for X_λ sufficiently large.

The normalization properties and the additivity property of the degree function thus defined follow immediately from the definition. The invariance under homotopies which are affine as well as homotopies of class $(S)_+$ asserted in Theorem 5 follows from the following two assertions:

PROPOSITION 12. *Each affine homotopy between two demicontinuous maps f and f_1 of class $(S)_+$ is a homotopy of class $(S)_+$.*

PROPOSITION 13. *If $\{f_t\}$ is a homotopy of class $(S)_+$ on* cl(G) *and suppose that $0 \notin f_t(\partial G)$ for any t in* $[0, 1]$. *Then there exists λ_0 in Λ such that for any t in* $[0, 1]$ *and all λ in Λ, $\lambda > \lambda_0$, we have $d(f_{t,\lambda}, G_\lambda, 0)$ well defined and independent of both t and λ.*

PROOF OF PROPOSITION 12. Suppose that $\{u_j\}$ is a sequence in $\mathrm{cl}(G)$ with $u_j \rightharpoonup u$, $\{t_j\}$ a sequence in $[0, 1]$ with $t_j \to t$, and such that

$$\overline{\lim}\left\{(1 - t_j)\left\langle f(u_j), u_j - u\right\rangle + t_j\left\langle f_1(u_j), u_j - u\right\rangle\right\} \leq 0.$$

Since f and f_1 are demicontinuous maps of class $(S)_+$, we know that

$$\underline{\lim}\left\langle f(u_j), u_j - u\right\rangle \geq 0, \qquad \underline{\lim}\left\langle f_1(u_j), u_j - u\right\rangle \geq 0.$$

Suppose without loss of generality that $t > 0$. Since

$$\underline{\lim}\,(1 - t_j)\left\langle f(u_j), u_j - u\right\rangle \geq 0$$

we see that

$$t\,\overline{\lim}\left\langle f_1(u_j), u_j - u\right\rangle = \overline{\lim}\left\{t_j\left\langle f_1(u_j), u_j - u\right\rangle\right\} \leq 0.$$

Since f is of class $(S)_+$, it follows that u_j converges strongly to u. Therefore

$$(1 - t_j)f(u_j) + t_j f_1(u_j) \rightharpoonup (1 - t)f(u) + t f_1(u),$$

and the proof of Proposition 12 is complete. Q.E.D.

PROOF OF PROPOSITION 13. Suppose that the assertion of Proposition 13 were false. Then by Proposition 11, for each X_λ in Λ, there would exist $X_\mu \supset X_\lambda$, a point u in ∂G_μ and a parameter value t in $[0, 1]$ such that

$$\left\langle f_t(u), u\right\rangle \leq 0, \qquad \left\langle f_t(u), v\right\rangle = 0 \quad (v \in X_\lambda).$$

For each λ in Λ we form a subset W_λ of the Cartesian product of $\partial G \times [0, 1]$, where

$$W = \left\{[u, t] \mid u \in \partial G, t \in [0, 1]; \left\langle f_t(u), u\right\rangle \leq 0; \left\langle f_t(u), v\right\rangle = 0\ (v \in X_\lambda)\right\}.$$

By our preceding consideration, each W_λ would be a nonempty subset of the bounded set $\partial G \times [0, 1]$, and the family $\{W_\lambda\}$ has the finite intersection property. If we take the closure of W_λ in the product of the weak topology on ∂G and the ordinary topology on $[0, 1]$ and denote this closure by w-cl(W_λ), then this set is compact in its appropriate topology. Hence

$$\bigcap_\lambda \text{w-cl}(W_\lambda) \neq \varnothing.$$

Let $[u_0, t_0]$ be a point of this intersection.

For an arbitrary point v of X, choose X_λ in Λ such that X_λ contains both u_0 and v. Since X is reflexive, we may choose a sequence $[u_j, t_j]$ in the corresponding W_λ such that $u_j \rightharpoonup u_0$, $t_j \to t$. For each index j, we have

$$\left\langle f_{t_j}(u_j), u_j\right\rangle \leq 0, \qquad \left\langle f_{t_j}(u_j), u_0\right\rangle = 0, \qquad \left\langle f_{t_j}(u_j), v\right\rangle = 0.$$

In particular it follows that

$$\overline{\lim}\left\langle f_{t_j}(u_j), u_j - u_0\right\rangle \leq 0.$$

If we apply the assumption that $\{f_t\}$ is a homotopy of class $(S)_+$, it follows that $u_j \to u_0$ and $f_{t_j}(u_j) \rightharpoonup f_{t_0}(u_0)$. Hence $u_0 \in \partial G$, and $\langle f_t(u_j), v\rangle \to \langle f_{t_0}(u_0), v\rangle = 0$. Since v is arbitrary, $f_{t_0}(u_0) = 0$ which contradicts our assumption that $0 \notin f_t(\partial G)$ for all t. Q.E.D.

We have now completed the proof of the existence of the degree in Theorem 5. It remains to prove the uniqueness of the degree in Theorem 4. We shall prove this fact in the following proposition.

PROPOSITION 14. *Let X be a reflexive Banach space and let d_1 be a degree function on the class F of demicontinuous mappings of class $(S)_+$ of the closures of bounded open sets G in X into X^*. Suppose that d_1 is invariant under the class of affine homotopies in F. Then d_1 coincides with the degree d defined above in terms of Galerkin approximants.*

PROOF OF PROPOSITION 14. Suppose the assertion were false. Then there would exist a bounded open set G in X, a demicontinuous map f of $\mathrm{cl}(G)$ into X^* of class $(S)_+$ and a point y_0 in X^*, $y_0 \notin f(\partial G)$ such that

$$d(f, G, y_0) \neq d_1(f, G, y_0).$$

We may assume without loss of generality that $y_0 = 0$.

By the definition of d in terms of Galerkin approximants, it follows from this assumption that for each λ in Λ, there would exist $X_\mu \supset X_\lambda$ such that $d_0(f_\mu, G_\mu, 0) \neq d_1(f, G, 0)$. We now apply the conclusion of Proposition 11 to the degree function d_1 on X and derive the existence of u in ∂G such that

$$\langle f(u), u \rangle \leq 0; \qquad \langle f(u), v \rangle = 0 \quad \text{for all } v \text{ in } X_\mu.$$

We define the subset V_λ of ∂G by

$$V = \{u \mid u \in \partial G;\ \langle f(u), u \rangle \leq 0;\ \langle f(u), v \rangle = 0 \text{ for all } v \text{ in } X_\lambda\}.$$

By the last paragraph, each V_λ is nonempty and the family $\{V_\lambda\}$ has the finite intersection property. By the weak compactness of w-cl(V_λ), it follows that $\bigcap_{\lambda \in \Lambda} \text{w-cl}(V_\lambda) \neq \varnothing$. Let u_0 be a point of this intersection.

Let X_λ be an element of Λ which contains both u_0 and a given arbitrary point v of X. By the reflexivity of X, there exists a sequence $\{u_j\}$ from the corresponding V_λ with $u_j \rightharpoonup u_0$. For each index j,

$$\langle f(u_j), u_j \rangle \leq 0; \qquad \langle f(u_j), u_0 \rangle = 0; \qquad \langle f(u_j), v \rangle = 0$$

and $u_j \in \partial G$. A fortiori $\overline{\lim}\langle f(u_j), u_j - u_0 \rangle \leq 0$, so that $u_j \to u_0$ and $f(u_j) \rightharpoonup f(u_0)$. Thus $u_0 \in \partial G$, and $\langle f(u_0), v \rangle = \lim \langle f(u_j), v \rangle = 0$. Since v was an arbitrary element of X, $f(u_0) = 0$, with $u_0 \in \partial G$. This is a contradiction, and the proof of Proposition 14 is complete.

Thus the proofs of Theorems 4 and 5 are complete.

If we seek to extend the degree function we have constructed to the broader class of pseudo-monotone mappings, we encounter the significant difficulty that the pseudo-monotone maps are not proper in the general case, and for an arbitrary bounded open subset G, it is not true for a pseudo-monotone map f of $\mathrm{cl}(G)$ into X^*, that $f(\partial G)$ is closed nor that $f(\mathrm{cl}(G))$ is closed. The latter will indeed be the case if G is convex, but is false in general otherwise. This necessitates a serious modification of the definition of a degree function if we are to obtain one for such mappings. We will not restate the formal definition of a degree function in its complete form to show the salient features of the new definition, but merely the modifications that are necessary.

DEFINITION 6. *We shall speak of a degree function in the extended (or weak) sense if the following modifications are made in the characterizing properties of the degree in Definition* 1.

(1) *The degree function* $d(f, G, y_0)$ *is to be defined only if* $y_0 \notin \mathrm{cl}(f(\partial G))$.

(2) *If* $d(f, G, y_0) \neq 0$, *then* $y_0 \in \mathrm{cl}(f(G))$.

(3) *The normalizing map* f_0 *is assumed to be proper on bounded sets.*

(4) *In the definition of the additivity property on domains, we must assume that*

$$y_0 \notin \mathrm{cl}\big(f(\mathrm{cl}(G)\setminus(G_1 \cup G_2))\big).$$

(5) *In the property of homotopy invariance, we must assume that if* Y *is a metric space, then a fixed ball* $B_r(y_t)$ *does not intersect* $f_t(\partial G)$ *for all* t *in* $[0, 1]$.

THEOREM 6. *There exists one and only one degree function in the extended sense on the class* F *of maps* $f\colon \mathrm{cl}(G) \to X^*$, *where* X *is a reflexive Banach space and the maps* f *are pseudo-monotone, with the degree invariant under affine homotopies and normalized by the duality mapping* J.

THEOREM 7. *The degree function of Theorem* 6 *is actually invariant under pseudo-monotone homotopies, where a homotopy* $\{f_t\}$ *is said to be pseudo-monotone if for a sequence* $\{u_j\}$ *converging weakly to* u *in* X *and a sequence* $\{t_j\}$ *converging to* t *in* $[0, 1]$ *for which* $\overline{\lim}\langle f_{t_j}(u_j), u_j - u\rangle \leq 0$, *we have* $\lim\langle f_{t_j}(u_j), u_j - u\rangle = 0$, *and if* u *lies in* $\mathrm{cl}(G)$, *then* $f_{t_j}(u_j) \rightharpoonup f_t(u)$.

We shall not give the detailed proofs of Theorems 6 and 7 here except to observe that in this case, the proof by Galerkin approximants does not suffice. We must instead approximate each pseudo-monotone mapping f by approximating mappings f_ε, $\varepsilon > 0$, in the class $(S)_+$ where $f_\varepsilon = f + \varepsilon J$ and J is the duality mapping of class $(S)_+$ which we have considered above. For each f which is pseudo-monotone and demicontinuous, the approximating map f_ε is demicontinuous and of class $(S)_+$. If $y_0 \notin \mathrm{cl}(f(\partial G))$, then for $\varepsilon > 0$ and sufficiently small, $y_0 \notin f_\varepsilon(\partial G)$. On the other hand, on any compact subinterval of $(0, \varepsilon_0)$, $\{f_\varepsilon\}$ is an affine homotopy of maps of class $(S)_+$. Therefore, by the invariance of the degree for maps of class $(S)_+$ under affine homotopies, $d(f_\varepsilon, G, y_0)$ will be independent of ε for $0 < \varepsilon < \varepsilon_0$. This common value, we designate as $d(f, G, y_0)$, and it will be our extended degree function for the class of pseudo-monotone mappings. The invariance under pseudo-monotone homotopies follows from the fact that for each $\varepsilon > 0$, $f_t + \varepsilon J$ yields a homotopy of class $(S)_+$. It does not follow in this case that each affine homotopy is pseudo-monotone (unless f and f_1 are bounded), but it is still the case that for each $\varepsilon > 0$, $\{f_t + \varepsilon J\}$ is an affine homotopy of maps of class $(S)_+$ and hence a homotopy of class $(S)_+$.

Restricted to the class of maps of class $(S)_+$, any degree in the extended sense just becomes a degree in the original sense of Definition 1 since all the maps and homotopies involved are proper. Thus an extended degree is unique by Theorem 4 on the dense subclass of maps of class $(S)_+$. By continuity of the degree under the homotopy $f + \varepsilon J$ as ε ranges over $[0, \varepsilon_0)$, we find the uniqueness of the extended degree on the class of pseudo-monotone mappings.

6. The degree for more general mappings of monotone type. There are a number of classes of mappings of monotone type with domain in a reflexive Banach space X and with values in the conjugate space X^* for which suitable extensions of the arguments which we developed above can be used to develop an existence and uniqueness theory for a suitably defined degree of mapping. To complete our presentation here, we consider the simplest and most generally useful of such classes, that of maps of the form $T + f$ with T maximal monotone and f bounded and of class $(S)_+$.

Let X be a reflexive Banach space. We consider maps T from X with values subsets of X^*. With each such map, we associate its graph $G(T)$ in $X \times X^*$, i.e.

$$G(T) = \{[u, w] \mid u \in X, w \in T(x)\}.$$

Then

The mapping T is said to be monotone if for any pair of elements $[u, w]$ and $[x, y]$ in $G(T)$, we have the inequality

$$\langle w - y, u - x \rangle \geqslant 0.$$

T is said to be maximal monotone if it is monotone and maximal in the sense of graph inclusion among monotone maps (multivalued) from X to X^*. An equivalent version of the last clause is that for any $[u_0, w_0]$ in $X \times X^*$ for which

$$\langle w_0 - y, u_0 - x \rangle \geqslant 0$$

for all $[x, y]$ in $G(T)$, we have $[u_0, w_0]$ in $G(T)$. If the Banach space X is reflexive, an equivalent statement is that the mapping $T + J$ has all of X^* as its range. (We assume as before that X has been given an equivalent norm in which the duality mapping J is bicontinuous and lies in the class $(S)_+$.)

Maximal monotone mappings occur in a number of useful and important contexts. First of all, all demicontinuous monotone maps of X into X^* are maximal monotone. Second, for any maximal monotone map T of X into the subsets of X^*, its inverse T^{-1} is maximal monotone from X^* to the subsets of X (this makes maximal monotonicity useful in the study of Hammerstein integral equations). Finally if φ: $X \to R \cup \{+\infty\}$ is a proper lower-semi-continuous convex function, its subgradient $\partial\varphi$ is maximal monotone, where

$$(\partial\varphi)(u) = \{w \mid w \in X^*; \text{For all } v \text{ in } X, \varphi(v) - \varphi(u) \geqslant \langle w, v - u \rangle\}.$$

This fact makes maximal monotone mappings a useful tool in the study of variational inequalities.

THEOREM 8. *Let X be a reflexive Banach space, T a maximal monotone mapping from X to 2^{X^*} with $0 \in T(0)$. Let G be a bounded open subset of X, and let f be a bounded mapping of* cl(G) *into X^* of class $(S)_+$. For each $\varepsilon > 0$, consider the* *generalized Yosida transformation* T_ε *corresponding to* T [3] *given by*

$$T_\varepsilon = (T^1 + \varepsilon J^{-1})^{-1}.$$

(A) *Suppose that for a given* y_0 *in* X^*, y_0 *does not lie in* $(T+f)(\partial G)$. *Then there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, y_0 *does not lie in* $(T_\varepsilon + f)(\partial G)$.

(B) *For each* $\varepsilon > 0$, *the mapping* $T_\varepsilon + f$ *of* $\mathrm{cl}(G)$ *into* X^* *is of class* $(S)_+$. *Hence for* $0 < \varepsilon < \varepsilon_0$, *the degree function* $d(T_\varepsilon + f, G, y_0)$ *is defined by the results of the preceding section. Moreover, for* $0 < \varepsilon < \varepsilon_1$, *the values of all these degrees coincide.*

DEFINITION 7. *We set the degree function* $d(T+f, G, y_0)$ *to be the common value of* $d(T_\varepsilon + f, G, y_0)$ *for* ε *sufficiently small.*

PROPOSITION 15. *Let* X *be a reflexive Banach space,* $\{T_t: 0 \leqslant t \leqslant 1\}$ *a family of maximal monotone maps from* X *to* 2^{X^*} *with* $0 \in T_t(0)$ *for all* t. *Let* J *be a duality mapping from* X *to* X^* *which corresponds to a norm on* X *in which both* X *and* X^* *are locally uniformly convex. Consider the following four conditions on the family* $\{T_t\}$:

(1) (*generalized pseudo-monotonicity*) *Suppose that for a sequence* $\{t_j\}$ *converging to* t *in* $[0, 1]$, *we have a sequence* $[u_u, w_j]$ *in* $G(T_{t_j})$ *with* u_j *converging weakly to* u *in* X, w_j *converging weakly to* w *in* X^*. *Suppose further* $\overline{\lim}\langle w_j, u_j\rangle \leqslant \langle w, u\rangle$. *Then* $w \in T_t(u)$, *and* $\langle w_j, u_j\rangle \to \langle w, u\rangle$.

(2) *Consider the mapping* φ *of* $X^* \times [0, 1]$ *into* X *given by*

$$\varphi(w, t) = (T_t + J)^{-1}(w).$$

Then φ *is continuous from* $X^* \times [0, 1]$ *to* X (*with both* X *and* X^* *given their strong topologies*).

(3) *For each* w *in* X^*, *the mapping* φ_w *of* $[0, 1]$ *into* X *given by*

$$\varphi_w(t) = (T_t + J)^{-1}(w)$$

is continuous from $[0, 1]$ *to the strong topology on* X.

(4) (*Strong lower-semi-continuity of* $G(T_t)$ *in* t) *Given* $[x, y]$ *in* $G(T_t)$ *and a sequence* $\{t_j\}$ *converging to* t *in* $[0, 1]$, *there exists a sequence* $[x_j, w_j]$ *with each* $[x_j, w_j]$ *in* $G(T_{t_j})$ *such that* x_j *converges strongly to* x *in* X, y_j *converges strongly to* y *in* X^*.

Then, conditions (1), (2), (3), *and* (4) *are mutually equivalent.*

DEFINITION 8. *A family* $\{T_t\}$ *of maximal monotone mappings which satisfies the mutually equivalent conditions* (1)–(4) *is called a pseudo-monotone homotopy of maximal monotone maps.*

PROOF OF PROPOSITION 15. We shall prove that (4) $\Rightarrow$ (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4). Obviously (2) $\Rightarrow$ (3).

PROOF THAT (4) IMPLIES (1). Let $[x, y]$ be an element of $G(T_t)$. For the sequence $\{t_j\}$ given in (1), choose a family $[x_j, y_j]$ as described in condition (4) with $[x_j, y_j]$ in $G(T_{t_j})$ and $[x_j, y_j]$ converging strongly to $[x, y]$. For each j, the monotonicity of T_{t_j} implies that

$$\langle w_j - y_j, u_j - x_j\rangle \geqslant 0.$$

Hence

$$\langle w, u\rangle \geqslant \overline{\lim}\,\langle w_j, u_j\rangle \geqslant \underline{\lim}\,\langle w_j, u_j\rangle \geqslant \lim\{\langle w_j, x_j\rangle + \langle y_j, u_j - x_j\rangle\}$$
$$= \langle w, x\rangle + \langle y, u - x\rangle.$$

Thus, $\langle w - y, u - x\rangle \geqslant 0$. Since this inequality holds for all $[x, y]$ in $G(T_t)$, the maximal monotonicity of T_t implies that $[u, w]$ lies in $G(T_t)$. On the other hand, setting $[x, y] = [u, w]$ in the inequality above, we see that

$$\langle w, u\rangle \geqslant \overline{\lim}\,\langle w_j, u_j\rangle \geqslant \underline{\lim}\,\langle w_j, u_j\rangle \geqslant \langle w, u\rangle.$$

Hence $\langle w_j, u_j\rangle \to \langle w, u\rangle$. Q.E.D.

PROOF THAT (1) IMPLIES (2). Let $u_j = (T_{t_j} + J)^{-1}(w_j)$ with w_j converging strongly to w in X^* and t_j converging to t in $[0, 1]$. Then

$$w_j = y_j + J(u_j)$$

for elements y_j in $G(T_{t_j})$. Since $[0, 0] \in G(T_{t_j})$ for each j, we have

$$\langle w_j, u_j\rangle = \langle y_j, u_j\rangle + \|u_j\|^2 \geqslant \|u_j\|^2.$$

Hence the sequence $\{u_j\}$ is bounded, as is the sequence $J(u_j)$. We wish to show that u_j converges strongly to $u = (T_t + J)^{-1}w$. To do so, it suffices to assume that u_j converges weakly to u, and show that u must be this given element and that the convergence is strong. We may also assume that $J(u_j)$ converges weakly to z in X^*, while $y_j = w_j - J(u_j)$ converges weakly to $w - z = y$.

For each j, we have

$$\langle w_j, u_j - u\rangle = \langle y_j, u_j - u\rangle + \langle J(u_j), u_j - u\rangle.$$

Since w_j converges strongly to w while $u_j - u$ converges weakly to 0, it follows that $\langle w_j, u_j - u\rangle \to 0$. On the other hand, since J is pseudo-monotone,

$$\underline{\lim}\,\langle J(u_j), u_j - u\rangle \geqslant 0.$$

Hence

$$\overline{\lim}\,\langle y_j, u_j - u\rangle \leqslant 0$$

i.e.

$$\overline{\lim}\,\langle y_j, u_j\rangle \leqslant \langle y, u\rangle.$$

If we apply condition (1), we find that $y \in T_t(u)$, that $\langle y_j, u_j\rangle \to \langle y, u\rangle$, i.e. $\langle y_j, u_j - u\rangle \to 0$. Hence

$$\langle J(u_j), u_j - u\rangle \to 0.$$

Since J is a map of class $(S)_+$ by hypothesis, u_j must converge strongly to u and $J(u_j)$ must converge strongly to $J(u)$. Hence $y = w - J(u)$, i.e. $w \in (T_t + J)(u)$, or $u = (T_t + J)^{-1}(w)$. Q.E.D.

PROOF THAT (3) IMPLIES (4). Let $[x, y] \in G(T_t)$. Then $y + J(x) \in (T_t + J)(x)$, i.e. $x = (T_t + J)^{-1}(y + J(x))$. If $\{t_j\}$ is a sequence converging to t in $[0, 1]$, let

$$x_j = (T_{t_j} + J)^{-1}(y + J(x)).$$

By condition (3), x_j converges strongly to x in X, so that $J(x_j)$ converges strongly to $J(x)$ in X^*. On the other hand, there exists y_j in $T(x_j)$ such that $y + J(x) = y_j + J(x_j)$. However, $y_j = y + J(x) - J(x_j)$ converges strongly to y in X^*. Q.E.D.

THEOREM 9. *Let X be a reflexive Banach space, G a bounded open subset of X, $\{f_t\colon 0 \leqslant t \leqslant 1\}$ a homotopy of class $(S)_+$ of maps of* cl(G) *into a bounded subset of X^*. Let $\{T_t\colon 0 \leqslant t \leqslant 1\}$ be a family of maximal monotone mappings of X into 2^{X^*} with $0 \in T_t(0)$ for all t. Suppose $\{T_t\}$ is a pseudo-monotone homotopy in the sense of Definition 1. Let $\{y(t)\colon t \in [0,1]\}$ be a continuous path in X^* such that $y(t) \notin (T_t + f_t)(\partial G)$ for all t in* $[0, 1]$. *Then*

(a) *For each $\varepsilon > 0$, $T_{t,\varepsilon} = (T_t^{-1} + \varepsilon J^{-1})^{-1}$ is a bounded pseudo-monotone mapping of X into X^*, while the family $\{T_{t,\varepsilon} + f_t\colon 0 \leqslant t \leqslant 1\}$ is a homotopy of class $(S)_+$ from* cl(G) *to X^*.*

(b) *There exists $\varepsilon_0 > 0$, such that for $0 < \varepsilon < \varepsilon_0$,*

$$y(t) \notin (T_{t,\varepsilon} + f_t)(\partial G),$$

$d(T_{t,\varepsilon} + f_t, G, y(t))$ is well defined and independent of t and ε.

THEOREM 10. *The degree function defined by Definition 1 on the class of maps of the form $T + f$ with T maximal monotone and f bounded and of class $(S)_+$ is a classical degree function with respect to the class of homotopies described in Theorem 2 with the canonical map $f_0 = J$.*

PROOF OF THEOREMS 8 AND 9. As stated, Theorem 9 includes Theorem 8 as a special case since the constant homotopy $T_t = T$ for all t is indeed pseudo-monotone in the sense of Definition 8. Hence it suffices to prove Theorem 9. The proof rests on the following auxiliary result.

PROPOSITION 16. *Let X be a reflexive Banach space, $\{T_t\colon t \in [0,1]\}$ a pseudo-monotone homotopy of maximal monotone maps from X to 2^{X^*} with $0 \in T_t(0)$ for all t. Let $\{u_j\}$ be a sequence in X converging weakly to u, and for sequences $\{t_j\}$ in* $[0, 1]$ *converging to t, $0 < \varepsilon_j$, $0 < \delta_j$, $\varepsilon_j \to 0$, $\delta_j \to 0$, let*

$$v_j = T_{t_j,\varepsilon_j}(u_j), \qquad z_j = T_{t_j,\delta_j}(u_j).$$

Suppose further that for another sequence $\{s_j\}$ in $[0, 1]$, *and for*

$$w_j = (1 - s_j)v_j + s_j z_j,$$

we have w_j converging weakly to w in X^*, while $\overline{\lim}\langle w_j, u_j\rangle \leqslant \langle w, y\rangle$.

Then $w \in T_t(u)$, and $\langle w_j, u_j\rangle \to \langle w, u\rangle$.

PROOF OF PROPOSITION 16. Since $v_j = T_{t_j,\varepsilon_j}(u_j)$, it follows that

$$v_j \in T_{t_j}\big(u_j - \varepsilon_j J^{-1}(v_j)\big).$$

Similarly, $z_j \in T_{t_j}(u_j - \delta_j J^{-1}(z_j))$. Since $0 \in T_{t_j}(0)$, and T_{t_j} is monotone,

$$\big(v_j, u_j - \varepsilon_j J^{-1}(v_j)\big) \geqslant 0, \qquad \big(z_j, u_j - \delta_j J^{-1}(z_j)\big) \geqslant 0.$$

Hence

$$\varepsilon_j \|v_j\|^2 \leq \langle v_j, u_j \rangle, \qquad \delta_j \|z_j\|^2 \leq (z_j, u_j).$$

If we multiply each of these inequalities by $(1 - s_j)$ and s_j, respectively and add, we obtain

$$(1 - s_j)\varepsilon_j \|v_j\|^2 + s_j \delta_j \|z_j\|^2 \leq \langle w_j, u_j \rangle \leq M.$$

Hence $(1 - s_j)\varepsilon_j \|v_j\|^2$ is bounded, as is $s_j \delta_j \|z_j\|^2$, so that

$$(1 - s_j)\varepsilon_j \|v_j\| \to 0, \qquad s_j \delta_j \|z_j\| \to 0.$$

Let $[x, y]$ be any element of $G(T_t)$. By the condition (4) for the pseudo-monotone homotopy $\{T_t\}$, for each j, there exist elements $[x_j, y_j]$ of $G(T_{t_j})$ with x_j converging strongly to x, y_j converging strongly to y. By the monotonicity of T_{t_j},

$$\left\langle v_j - y_j, u_j - \varepsilon_j J^{-1}(v_j) - x_j \right\rangle \geq 0, \qquad \left\langle z_j - y_j, u_j - \delta_j J^{-1}(z_j) - x_j \right\rangle \geq 0.$$

Hence,

$$\langle v_j - y_j, u_j - x_j \rangle \geq \varepsilon_j \left\langle v_j - y_j, J^{-1}(v_j) \right\rangle \geq -\|y_j\| \varepsilon_j \|v_j\|,$$

and

$$\langle z_j - y_j, u_j - x_j \rangle \geq \delta_j \left\langle z_j - y_j, J^{-1}(z_j) \right\rangle \geq -\|y_j\| \delta_j \|z_j\|.$$

If we multiply these inequalities by $(1 - s_j)$ and s_j, respectively, and add, we get

$$\langle w_j - y_j, u_j - x_j \rangle \geq -M\left\{(1 - s_j)\varepsilon_j \|v_j\| + s_j \delta_j \|z_j\|\right\} \to 0.$$

Thus

$$\begin{aligned} \langle w, u \rangle &\geq \overline{\lim} \langle w_j, u_j \rangle \geq \underline{\lim} \langle w_j, u_j \rangle \geq \underline{\lim} \left\{ \langle w_j - y_j, +x_j \rangle + \langle y_j, +u_j \rangle \right\} \\ &= \langle w - y, x \rangle + \langle y, u \rangle. \end{aligned}$$

Hence

$$\langle w - y, u - x \rangle \geq 0 \qquad ([x, y] \text{ in } G(T_t)),$$

and by the maximal monotonicity of T_t, $w \in T_t(u)$. Substituting $[u, w]$ for $[x, y]$ in the preceding chain of inequalities, we see that

$$\langle w, u \rangle \geq \overline{\lim} \langle w_j, u_j \rangle \geq \underline{\lim} \langle w_j, u_j \rangle \geq \langle w, u \rangle.$$

Thus, finally, $\langle w_j, u_j \rangle \to \langle w, u \rangle$. Q.E.D.

PROOF OF THEOREM 9 COMPLETED. By Proposition 12, it follows that if $\{T_t\}$ is a pseudo-monotone homotopy, then T_t^{-1} is a pseudo-monotone homotopy. Similarly, if $\{T_t^{-1}\}$ is a pseudo-monotone homotopy and if $\{g_t\}$ is a pseudo-monotone homotopy of bounded monotone maps, then $\{T_t^{-1} + g_t\}$ is a pseudo-monotone homotopy. In particular, $\{T_t^{-1} + \varepsilon J^{-1}\}$ is a pseudo-monotone homotopy, and $T_{\varepsilon,t} = (T_t^{-1} + \varepsilon J^{-1})^{-1}$ is also a pseudo-monotone homotopy. Hence $T_{\varepsilon,t} + f_t$ is a pseudo-monotone homotopy of class $(S)_+$ for each $\varepsilon > 0$. Thus part (a) holds.

Suppose the conclusion of part (b) were false. Then there would exist a sequence $\{t_j\}$ in $[0, 1]$ and two sequences of positive numbers $\varepsilon_j, \delta_j \to 0$ and points

u_j on ∂G, such that for suitable s_j in [0, 1]

$$y_{t_j} = (1 - s_j)T_{t_j,\varepsilon_j}(u_j) + s_j T_{t_j,\delta_j}(u_j) + f_{t_j}(u_j).$$

We may assume that t_j converges to t, and that y_{t_j} converges strongly to y_t. If we set $v_j = T_{t_j,\varepsilon_j}(u_j)$, $z_j = T_{t_j,\delta_j}(u_j)$, and $w_j = y_{t_j} - f_{t_j}(u_j)$, we may assume that u_j converges weakly to u and that w_j converges weakly to w. Moreover,

$$\overline{\lim}\, \langle w_j, u_j - u \rangle = \lim \langle y_{t_j}, u_j - u \rangle - \lim \left\langle f_{t_j}(u_j), u_j - u \right\rangle \leq 0$$

since

$$\underline{\lim}\, \left\langle f_{t_j}(u_j), u_j - u \right\rangle \geq 0$$

by the pseudo-monotonicity of the homotopy $\{f_t\}$. If we apply Proposition 2, it follows that $w \in T_t(u)$, and that $\langle w_j, u_j - u \rangle \to 0$. Hence

$$\left\langle f_{t_j}(u_j), u_j - u \right\rangle \to 0,$$

and since the homotopy $\{f_t\}$ is of class $(S)_+$, u_j converges strongly to u, and $f_{t_j}(u_j)$ converges weakly to $f_t(u)$. Thus u lies in ∂G, $y_t - f_t(u) = w \in T_t(u)$, i.e. $y_t \in (T_t + f_t)(\partial G)$, which contradicts the hypothesis. Q.E.D.

THEOREM 11. *The degree function described by Theorem 3 can be extended to the class of mappings of the form $T + f$, with f pseudo-monotone and bounded by letting $d(T_t + f, G, y_0) = \lim_{\delta \to 0} d(T + f + \delta J, G, y_0)$, where the mapping $f + \delta J$ is of class $(S)_+$. This extension has properties similar to the corresponding extension of the degree function to pseudo-monotone mappings in* §5.

We turn now to the uniqueness result for the degree function over the class $T + f$, with T maximal monotone, f of class $(S)_+$ and bounded.

THEOREM 12. *Let X be a reflexive Banach space. Then there exists exactly one degree function on the class of maps $T + f$, with T maximal monotone and f bounded and of class $(S)_+$, which is normalized by J and such that the degree is invariant under all affine homotopies of the form*

$$(1 - t)(T + f) + tf_1$$

with T maximal monotone, f and f_1 of class $(S)_+$.

PROOF OF THEOREM 12. Let d_1 be such a degree function. When restricted to the maps of class $(S)_+$, it coincides with the unique degree function on that class. Hence, we must show that this unique identification persists when we pass to the broader class $(T + f)$.

Suppose $y_0 \notin (T + f)(\partial G)$. Consider the affine homotopy between $(T + f)$ and the approximating map $(T_\varepsilon + f)$ for small $\varepsilon > 0$ which we employed above in proving the existence of the degree. If $d_1 \neq d$, then for arbitrary small $\varepsilon > 0$, we can find u_ε on ∂G and t_ε in [0, 1] such that

$$(1 - t_\varepsilon)T(u_\varepsilon) + t_\varepsilon T_\varepsilon(u_\varepsilon) + f(u_\varepsilon) \ni y_0.$$

We may choose a sequence u_j in ∂G, $u_j \rightharpoonup u$, $t_j \rightharpoonup t$, $\varepsilon_j \to 0$, $f(u_j) \rightharpoonup v_0$. For each j, we have $w_j \in T(u_j)$ such that

$$(1 - t_j)w_j + t_j T_{\varepsilon_j}(u_j) + f(u_j) = y_0.$$

We now apply a variant of the argument used in the proof of Proposition 16. If $v_j = T_{\varepsilon_j}(u_j)$, then

$$v_j \in T\big(u_j - \varepsilon_j J^{-1}(v_j)\big).$$

Similarly, $w_j \in T(u_j)$. Hence

$$\varepsilon_j \| v_j \|^2 \leqslant \langle v_j, u_j \rangle, \qquad 0 \leqslant \langle w_j, u_j \rangle.$$

If we multiply these inequalities by $(I - t_j)$ and t_j, respectively, and add, we obtain

$$t_j \varepsilon_j \| v_j \|^2 \leqslant \big\langle y_0 - f(u_j), u_j \big\rangle \leqslant M.$$

Hence $t_j \varepsilon_j \| v_j \|^2$ is bounded, so that $t_j \varepsilon_j \| v_j \| \to 0$.

Let $[x, y]$ be any element of $G(T)$. Then

$$\big\langle v_j - y, u_j - \varepsilon_j J^{-1}(v_j) - x \big\rangle \geqslant 0, \qquad \langle w_j - y, u_j - x \rangle \geqslant 0.$$

Thus,

$$\langle v_j - y, u_j - x \rangle \geqslant \big\langle v_j - y, \varepsilon_j J^{-1}(v_j) \big\rangle \geqslant \big\langle -y, \varepsilon_j J^{-1} v_j \big\rangle \geqslant -\varepsilon_j \| v_j \| \, \| y \|.$$

If we multiply the inequalities by t_j and $(1 - t_j)$, respectively, and add, we obtain

$$\big\langle y_0 - f(u_j) - y, u_j - x \big\rangle \geqslant -\varepsilon_j t_j \| v_j \| \, \| y \|,$$

where the term on the right approaches zero as $j \to +\infty$. Hence, if $z_j = y_0 - f(u_j)$, then $z_j \rightharpoonup z = y_0 - v$, and

$$\underline{\lim} \langle z_j - y, u_j - x \rangle \geqslant 0.$$

On the other hand, since $z_j + f(u_j) = y_0$, we have

$$\langle z_j, u_j - u \rangle + \big\langle f(u_j), u_j - u \big\rangle = \langle y_0, u_j - u \rangle,$$

i.e.

$$\overline{\lim} \langle z_j, u_j - u \rangle + \underline{\lim} \big\langle f(u_j), u_j - u \big\rangle \leqslant 0.$$

Since f is of class $(S)_+$, $\underline{\lim} \langle f(u_j), u_j - u \rangle \geqslant 0$. Hence

$$\overline{\lim} \langle z_j, u_j - u \rangle \leqslant 0, \qquad \overline{\lim} \langle z_j, u_j \rangle \leqslant \langle z, u \rangle.$$

Thus

$$\langle z, u \rangle \geqslant \overline{\lim} \langle z_j, u_j \rangle \geqslant \underline{\lim} \langle z_j, u_j \rangle \geqslant \langle y, u - x \rangle + \langle z, x \rangle.$$

It follows that $\langle z - y, u - x \rangle \geqslant 0$ for all $[x, y]$ in $G(T)$. By the maximal monotonicity of T, $z \in T(u)$. Replacing $[x, y]$ by $[u, z]$, we find that $\langle z_j, u_j \rangle \to \langle z, u \rangle$. Hence

$$\overline{\lim} \big\langle f(u_j), u_j - u \big\rangle \leqslant \overline{\lim} \langle z_j, u - u_j \rangle = 0.$$

Since f is of class $(S)_+$, $u_j \to u$ so that u lies on ∂G, and $f(u_j) \to f(u)$. Thus $f(u) = y_0 - z$, or in other words, $y_0 \in (T+f)(u)$. This contradicts the assumption that $y_0 \notin (T+f)(\partial G)$. The contradiction is to the assumption that the two degrees are different for $(T+f)$. Hence the two degrees coincide. Q.E.D.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637