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Subnormal operators, by John B. Conway, Research Notes in Math., 51, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1981, xvii + 476 pp., $24.95. ISBN 0-8218-2184-9

The theory of subnormal operators is the symbiotic interplay between operator theory and the theory of rational approximation. To see the connection, recall first some definitions. A (bounded linear) operator T on a (complex) Hilbert space $\mathcal{H}$ is called normal if T commutes with its adjoint, T^*. The operator T is called subnormal if it is possible to imbed $\mathcal{H}$ in a larger Hilbert space $\mathcal{K}$ and to find a normal operator N on $\mathcal{K}$ leaving $\mathcal{H}$ invariant (i.e., $N\xi \in \mathcal{H}$, $\xi \in \mathcal{H}$) such that T is the restriction of N to $\mathcal{H}$. If T is any operator, and if K is its spectrum (a compact subset of the plane), then in a familiar way one may form $r(T)$ where r is a rational function with poles located in the complement of K. One says that T is rationally cyclic if there is a vector ξ_0 in the Hilbert space of T, $\mathcal{H}$, such that $\mathcal{H}$ is the closed linear span of the vectors $r(T)\xi_0$ where r runs through the rational functions with poles off K. The starting point of the theory of subnormal operators, and its link with approximation theory, is the observation that if T is a rationally cyclic subnormal operator on $\mathcal{H}$, then there is a finite positive measure μ on the spectrum, K, of T and there is a Hilbert space isomorphism V from $\mathcal{H}$ onto $R^2(K, \mu)$, the closure in $L^2(K, \mu)$ of the space of rational functions with poles off K, such that $(VTV^{-1}\xi)(z) = z\xi(z)$ for all $\xi \in R^2(K, \mu)$. This transform of T by V is usually denoted by M_z. While a lot of attention has been paid to general subnormal operators, it is fair to say that most of the work in the subject to date, and the deepest, has been devoted to understanding M_z on $R^2(K, \mu)$.

At first glance, it may appear that the representation of a rationally cyclic subnormal operator in this fashion makes the study of subnormal operators trivial. After all, how could anything so concrete be inscrutable? It turns out, however, that the representation is no panacea. Some of the most fundamental facts require deep analysis and many basic problems remain to be solved. For example, the problem of deciding if a subnormal operator has a proper invariant subspace was settled (in the affirmative) only in 1978 by Scott Brown **[6]**. His work, in turn, was based on Sarason's penetrating analysis **[19]** of $P^\infty(\mu)$, the weak-* closure of the space of polynomials in $L^\infty(\mu)$, where μ is a compactly supported measure in the plane. Except for (essentially only) one example, the invariant subspace structure of M_z on $R^2(K, \mu)$ is a complete mystery. The exception is when K is the closed unit disc and μ is arclength measure on the boundary. The description of the invariant subspaces of M_z in this context is Beurling's famous theorem **[5]** which asserts that each invariant subspace is of the form $\theta R^2(K, \mu)$ where θ is a so-called inner function. Beurling's result is definitive .because the structure of inner functions is so completely well understood. There is also an important generalization of

Beurling's theorem to the setting where K is the closure of a finitely connected domain with smooth boundary and μ is arclength measure on the boundary. This generalization describes the subspaces of $R^2(K, \mu)$ that are invariant under every operator $r(M_z)$ where r runs through the rational functions with poles off K. Even in this context, however, the larger class of subspaces that are invariant simply under M_z alone remains to be determined (cf., however, [**18**]). Perhaps the most frustrating example that has defied analysis is the "simple" operator M_z on $R^2(K, \mu)$ where K is the closed unit disc and μ is planar Lebesgue measure (cf. [**14**]). For another example where basic problems have received a lot of attention but on which much remains to be done, consider the problem of determining the fine structure of the spectrum of M_z on $R^2(K, \mu)$. The key analytic ingredient which requires attention is the set of so-called bounded point evaluations, i.e. the points λ in K with the property that there is a constant M such that for every rational function r with poles off K, $|r(\lambda)| \leq M\|r\|$. The conjugates of these points coincide with the eigenvalues of the adjoint of M_z and, generally, it is a very difficult task to identify them explicitly in terms of the measure μ and the geometry of K.

The author asserts: "This book is actually somewhere between a research monograph and a textbook... After reading this book, there are virtually no papers on subnormal operators that the reader will not be able to attack with confidence." These are very accurate statements indeed. The prerequisites for reading *Subnormal operators* are minimal. They consist of what is usually meant by a first year graduate course in real and complex analysis followed by approximately a semester of judiciously chosen topics in functional analysis. Readers should know the basic facts about Banach spaces and the geometry of Hilbert space; they should know what operators are, but they need not know the spectral theorem; and they should know the rudiments of the Gelfand theory of commutative Banach algebras. The subject, subnormal operators, is an eclectic one and unaided students of it surely would find themselves awash in ancillary technicalities. However, the author has collected all the necessary tools with the meticulous care that is so characteristic of his earlier text [**8**]. On the other hand, the book is quite up to date, and anyone who completes it will indeed be well prepared to explore the frontiers of the subject. A little over 7% of the 258 items in the bibliography are listed as preprints and many of the results which they contain appear in the text. In numerous instances, the author gives "state of the art" proofs, as we shall see momentarily, and references to alternative proofs or original sources are given whenever they are appropriate. In addition to numerous exercises, some of which are incorporated in the text, the book contains a rather extensive list of open problems. These should keep investigators in the field busy for some time to come.

The first two chapters are preparatory in nature. Chapter I is a potpourri of basic facts needed in the later parts of the text. Topics include: applications of Runge's theorem, one of the main tools of the subject; the Riesz functional calculus; a review of compact operators; the Krein-Smulian theorem; the structure of an isometry; and the basics of Fredholm theory. Chapter II is a complete treatment of the spectral theorem for normal operators, including multiplicity theory. It is one of the pet peeves of the reviewer that so many texts in functional analysis treat the spectral theorem as an arcane topic and that multiplicity theory, which really is the sine qua non of the subject, is almost never mentioned. Teachers of functional analysis would be well advised to use the 48 pages that comprise this chapter as the text for their presentation of the spectral theorem. It is lucid, down to earth, and complete.

The author gets down to business in Chapter III and covers a number of basic topics about subnormal operators. They are introduced along with several relatives. Halmos's intrinsic characterization of subnormal operators is presented together with a number of variations. The representation of a rationally cyclic subnormal operator as M_z on $R^2(K, \mu)$ is proved and immediate consequences are drawn. Considerable attention is devoted to examples in this chapter. The unilateral shift, which is unitarily equivalent to M_z on $R^2(K, \mu)$ where, K is the closed unit disc and μ is Lebesgue measure on the boundary, is studied in detail and Beurling's theorem is proved. Also, shifts of higher multiplicity are analyzed briefly. Weighted shifts form a rich source of examples in operator theory generalized, and the theory of subnormal operators, in particular, is no exception. Characterizations of subnormal weighted shifts, both unilateral and bilateral, are presented. Bounded point evaluations and their role in the spectral theory of subnormal operators are discussed. A special, but very important class of subnormal operators, the Bergman operators, are discussed in great detail. These are defined as follows. Let G be a bounded open set in the plane and let $L_a^2(G)$ denote the subspace of functions in $L^2(G)$ (area measure) that are analytic. This space is called the Bergman space after S. Bergman who first studied such spaces systematically. The operator of multiplication by z on $L_a^2(G)$ is called the Bergman operator associated with G. Bergman operators are subnormal, both whether or not one is rationally cyclic depends on the geometry of G. The author discusses this dependence and presents the recent results on the spectrum of Bergman operators that he obtained with Axler and McDonald [**2**]. The commutant of a subnormal operator is discussed, as is the C^*-algebra generated by a subnormal operator. The chapter concludes with a discussion of three types of equivalence between operators: unitary equivalence, similarity, and quasisimilarity. The first two are well-known concepts. As for the third, one says that operators T_1 and T_2 are quasisimilar if there are operators X and Y with zero kernels and dense ranges such that $T_1X = XT_2$ and $YT_1 = T_2Y$. For normal operators, these three notions coincide, thanks to the Putnam-Fuglede theorem. However, for subnormal operators, the three notions are distinct. Quasisimilarity is a refractory concept, and the study of it has only just begun. One of the more piquant questions about quasisimilarity and subnormal operators cited by the author is this: Do quasisimilar cyclic operators have naturally isomorphic commutants? Surely the answer is yes, but reflection will reveal that the problem is not easy.

Chapter IV surveys the basic facts about Hardy spaces on the unit disc. These are included for completeness and to help the uninitiated. The principal points covered are: boundary values of H^p-functions; the inner-outer factorization of H^p-functions and the fine structure of inner functions (this gives the lattice structure of the invariant subspaces for the unilateral shift); Szegös' theorem; and the F. and M. Riesz theorem. By and large, what is included here is standard fare, but what is nice from the pedagogical perspective is the use of the maximal function to study the boundary behavior of H^p functions and to prove the F. and M. Riesz theorem. Such techniques and the branch of harmonic analysis from which they come are playing an increasingly important role in certain areas of operator theory (cf. [**20**]) and operator theorists are well advised to learn them.

Hyponormal operators are the subject of Chapter V. An operator T is hyponormal if $T^*T \geqslant TT^*$. Every subnormal operator is hyponormal, but hyponormal operators form a much larger class. Perhaps a historical note should be inserted here. Halmos introduced subnormal operators in [**10**], but the operators he called subnormal have since come to be called hyponormal; what we now call a subnormal operator, Halmos called a completely subnormal operator. (This has to do with his intrinsic characterization.) In [**12**], Halmos adopted the term 'subnormal' for subnormal operators, but in [**11**] he called an operator T with the property that $T^*T - TT^*$ either positive semidefinite or negative semidefinite seminormal. This term is still used from time to time. The more popular term 'hyponormal' was coined by Berberian [**3**]. Until recently, the theories of hyponormal operators and subnormal operators have evolved somewhat separately, concentrating on different problems and using rather different techniques. (See [**7** and **16**] for accounts of the theory of hyponormal operators.) However, there are several results about hyponormal operators that have proved useful in the theory of subnormal operators. Curiously, no one has found simpler proofs in the subnormal case. The main result presented in the text is Berger-Shaw theorem [**4**] that asserts that if T is hyponormal on a Hilbert space $\mathcal{H}$, then the trace of $T^*T - TT^*$ is bounded by n/π times the area of the spectrum of T. Here, n is the smallest number of vectors, $\xi_1, \ldots, \xi_n$ in $\mathcal{H}$, needed so that $\mathcal{H}$ is spanned by $r(T)\xi_i$ as r ranges over the rational functions with poles off the spectrum of T and i runs from 1 to n. This is indeed a deep theorem. As a consequence, we find that the spectrum of an arbitrary hyponormal operator that is not normal has positive area, a fact proved earlier by Putnam [**17**] using different, but still difficult, techniques. The proof that the author presents of the Berger-Shaw theorem is Voiculescu's [**22**] which rests on the notion of quasi-triangularity, another concept introduced by Halmos [**13**].

The sixth chapter is an introductory vignette on the theory of function algebras with special emphasis on their applications to rational approximation theory and operator theory. Most of the results presented here may be found either in [**9**] or in [**21**]. The main reason for including Chapter VI is to pave the way to Chapter VII whose primary purpose is to expose Sarason's theorem characterizing $P^\infty(\mu)$. The author presents Scott Brown's proof which avoids the theory of analytic capacity. In addition to being of great interest in its own right, it is the stepping stone to Scott Brown's invariant subspace theorem, the functional calculus and spectral mapping theorem for subnormal operators, and other developments. These things are taken up in the final chapter, Chapter VIII.

In an epilogue, the author lists some topics he might have taken up, but didn't. From this reviewer's perspective, it is too bad that two of these weren't covered. The first is the paper by Olin and Thompson [**15**] which shows that a subnormal operator is reflexive. This would have complemented nicely Scott Brown's theorem. The second is the extension of Beurling's theorem to multiply connected domains mentioned earlier. This would certainly help the reader interested in studying the paper of Abrahamse and Douglas [**1**] as well as other work in the area which is closely tied to hypodirichlet algebras. These omissions are matters of taste; in no material way do they detract from the overall high quality of the book.

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