The book begins with basic facts about Banach spaces, convolution structures, approximate identities and then carefully prepares three of the most useful tools of modern analysis—the Fourier transform, harmonic functions, and the interpolation of operators. The penultimate chapter is an introduction to the maximal function and an application of these ideas to ergodic theory (in the style of Cotlar's famous paper [2]). The final chapter begins with conjugate functions, Riesz transforms, and then goes on to simpler sorts of Calderón—Zygmund kernels (they satisfy the Dini condition).

This book grew out of a series of lectures Sadosky gave at the Universidad Central de Venezuela and the presentation of many of the topics shows the influence of the texts mentioned above. Unfortunately for the student, the material is given only a formal motivation (study A in order to learn A), and there are no exercises.

One final remark. Both books are produced from camera-ready copy. Even though the manuscripts were very carefully prepared on high-quality typewriters, we should think that for the price, the reader could expect the much more readable and attractive typesetting.

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Function theory of several complex variables, by Steven G. Krantz, John Wiley & Sons, New York, 1982, xiii + 437 pp., $39.95. ISBN 0-4710-9324-6

1. Levi problem. Many interesting results in complex analysis are concerned with the existence of analytic functions with prescribed properties. Most of these apply to "pseudoconvex" domains: this is a holomorphically invariant geometric criterion generalizing convexity. One such result is the solution of the so-called Levi problem: *Every pseudoconvex domain $\Omega \subset \mathbf{C}^n$ is a domain of* *holomorphy* (i.e. there exists an analytic function on Ω which cannot be continued over any point $p \in \partial\Omega$).

All domains $\Omega \subset \mathbf{C}^1$ are pseudoconvex (trivially). Further, each domain $\Omega \subset \mathbf{C}^1$ is a domain of holomorphy. (This is seen because for each $p \in \partial\Omega$ the function $(z-p)^{-1}$ has a natural barrier at p. By an application of the Baire Category Theorem, there is a single function $f(z)$ on Ω with a barrier at each $p \in \partial\Omega$.)

The Levi problem becomes nontrivial for domains $\Omega \subset \mathbf{C}^n$ when $n \geqslant 2$. A little experimentation will show that there is no "natural" singularity generalizing $(z-p)^{-1}$ to 2 or more variables. In fact there can be no singularity at all at any "inner" boundary point of the domain

$$\Omega = \{(z, w) \in \mathbf{C}^2 \colon 1 \leqslant \max(|z|, |w|) < 2\}.$$

This is because each function on Ω may be continued analytically to $\tilde{\Omega} = \{(z, w) \in \mathbf{C}^2 \colon \max\{|z|, |w|\} < 2\}$. (Consider the Laurent expansion in z and w of a function analytic on Ω.)

A widely used method for existence theorems is to proceed via the operator

$$\bar{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j.$$

A function f is analytic if and only if $\bar{\partial} f = 0$, since this system contains the Cauchy-Riemann equations in each variable. The "$\bar{\partial}$-method" might be described loosely as follows. First, one must find a norm $\| \quad \|_0$ on functions and a norm $\| \quad \|_1$ on (0, 1)-forms such that for each (0, 1)-form μ with $\bar{\partial}\mu = 0$, there is a function h satisfying $\bar{\partial} h = \mu$ and

$$\|h\|_0 \leqslant C\|\mu\|_1. \tag{1}$$

For any function f on Ω, the new function $F = f - g$ is analytic whenever $\bar{\partial} f = \bar{\partial} g$. Note that if we choose f to be almost analytic (i.e. $\bar{\partial} f$ is small), then the correction term g may be taken to be small by (1). The second part of the $\bar{\partial}$-method, then, is to arrange that the function $f(z)$ has the desired properties and that these are not lost after the addition of a correction.

This method yields, among many other existence theorems, a solution of the Levi problem. Thus it turns out that the study of domains of holomorphy (trivial for $n = 1$) leads to the interplay of the geometric property of pseudoconvexity and the analytic nature of $\bar{\partial}$.

2. Krantz's book. Analytic continuation and the Levi problem form the core around which much of Krantz's book is organized. The first seven chapters cover the basic material on analytic functions. This includes thorough treatments of convexity and subharmonicity and a solution of $\bar{\partial}$ via Hörmander's weighted L^2-estimates. The general framework for all of this is contained in Cartan's Theorems A and B, which are discussed at length; but the proof, being sheaf-theoretic and somewhat in a different direction, is omitted. A drawback is that Chapter 1 jumps immediately into some integral formulas which are rather advanced and specialized; some of this might be skipped upon a first reading.

The last three chapters cover special topics: boundary values, integral formulas, and holomorphic mappings, which are discussed below.

Krantz writes from the point of view of the analyst: much attention is given to precise smoothness and integrability properties of functions, and many techniques from real analysis are introduced and used. In fact, the real variable techniques are emphasized, in view of their greater generality. The Introduction reveals the missionary spirit of the book. It is designed to take the student familiar with analytic functions of a single variable and lure him into the subject of several variables. It is written in a chatty, informal style and is quick to answer the "trivial" questions a student might ask.

One of the remarkable features of this book is that it contains a large number of Exercises and Problems. In fact this seems to be the first treatment of the subject to have a significant number of problems at all. This is particularly important since the subject abounds with abstract theorems and has relatively few concrete (and nontrivial) worked examples. Perhaps this is related to the fact that there are rather few connections with applied mathematics. Several of Krantz's problems are difficult; the student will probably have to look for hints in the references cited.

To be sure, this text is no replacement for the classic book of Hörmander, which is unsurpassed for its power and elegance. But Hörmander demands considerable sophistication from his reader, and the student needs a book like Krantz's, which is written as a text with explanations and exercises.

3. Boundary values. Many properties of analytic functions are really just properties of harmonic (or subharmonic) functions. If u is a bounded harmonic function on the unit ball $\mathbf{B}^n = \{|z|^2 < 1\} \subset \mathbf{C}^n$, then it is the Poisson integral of a function $u^* \in L^\infty(\partial \mathbf{B}^n)$. Further,

$$\lim_{\substack{\zeta \to z \\ \zeta \in C_z}} u(\zeta) = u^*(z)$$

holds for almost every $z \in \partial \mathbf{B}^n$, with the limit being taken through a truncated cone $C_z \subset \mathbf{B}^n$ with its apex at $z \in \partial \mathbf{B}^n$.

But a function satisfying $\bar{\partial} f = 0$ in fact satisfies n independent equations and is thus the solution of an over-determined system of differential equations. In particular f is also harmonic on smaller subsets of $\mathbf{B}^n$ and boundary limits will exist on sets with much finer structure. For instance, if $a = (a_1, \ldots, a_n) \in \partial \mathbf{B}^n$, then $F(\zeta) = f(\zeta a_1, \ldots, \zeta a_n)$ is analytic and bounded on the unit disk $\{|\zeta| < 1\}$. Thus there are nontangential boundary limits at almost every point of the circle $\{(e^{i\theta} a_1, \ldots, e^{i\theta} a_n),\ 0 \leq \theta < 2\pi\} \subset \partial \mathbf{B}^n$. Similar results hold for more general curves and in more general domains, and the boundedness condition may be relaxed. Further, it has been shown that the nontangential approach region C_z may be replaced by a larger "admissible" region A_z, which is partially tangential to the boundary. Although many generalizations have been obtained, there remains the question of just how small the set

$$\left\{ z \in \partial \mathbf{B}^n \colon \lim_{r \to 1} f(rz) \text{ does not exist} \right\}$$

really is.

Another way that the overdetermined nature of the equation $\bar{\partial}f = 0$ expresses itself when $n \geq 2$ is that there is the induced boundary operator $\bar{\partial}_b$ on $\partial \mathbf{B}^n$. For instance, at the point $(1, 0, \ldots, 0) \in \partial \mathbf{B}^n$, $\bar{\partial}_b = \sum_{j=2}^n (\partial/\partial \bar{z}_j) d\bar{z}_j$. Boundary values and $\bar{\partial}_b$ are related by the following result.

THEOREM. *A function* $f^* \in L^\infty(\partial \mathbf{B}^n)$ *is the nontangential boundary value of an analytic function on* $\mathbf{B}^n$ *if and only if* $\bar{\partial}_b f^* = 0$.

Thus whether or not f^* is the boundary value of a holomorphic function is a local condition if $n \geq 2$; it is not local if $n = 1$. The operator $\bar{\partial}_b$ on $\partial \mathbf{B}^2$ is (after a change of coordinates) the famous "unsolvable" operator of H. Lewy; the fact that $\bar{\partial}_b u = h$ is not locally solvable on $\partial \mathbf{B}^2$ is closely related to the Theorem above.

By now we have probably given the misleading impression that boundary values are thoroughly good and that the only problem is to find proofs of these good properties. Let us give an example of the opposite situation. For functions of one variable, a special role is played by the *inner functions*, i.e. the bounded functions on the unit disk for which $|f^*(e^{i\theta})| = 1$ for a.e. θ. The following heuristic argument shows that an inner function f on $\mathbf{B}^n$, $n \geq 2$, should be constant. If $|f^*| = 1$, then $\bar{f}^* = 1/f^*$, and so

$$\bar{\partial}_b(\overline{f^*}) = \bar{\partial}_b(1/f^*) = -\frac{\bar{\partial}_b f^*}{(f^*)^2} = 0.$$

It follows that $\bar{\partial}_b \operatorname{Re} f^* = \bar{\partial}_b(f^* + \bar{f}^*) = 0$ and so $\operatorname{Re} f^*$ is the boundary value of an analytic function, and is thus constant. There are also completely different but equally convincing heuristic arguments to show that inner functions are constant. Thus it was quite a surprise when it was recently shown[1] that there *do exist* nonconstant inner functions: the boundary behavior can be much more complicated than was previously thought.

4. Cauchy integral formula. Let $D \subset \mathbf{C}^1$ be a domain with reasonable boundary. For any function $g \in C^1(\bar{D})$ we have the Cauchy-Green formula

$$g(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{g(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2\pi i}\iint_D \frac{\partial g(\zeta)}{\partial \bar{\zeta}}\,\frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} \tag{2}$$

for all $z \in D$. If we abbreviate this as

$$g(z) = K_1(g, D) + K_2(\bar{\partial} g, D), \tag{3}$$

it is apparent that

(a) the forms appearing in K_1 and K_2 are independent of D,

(b) the kernel K_1 is holomorphic in z for $z \neq \zeta$,

(b′) K_2 solves $\bar{\partial}$, i.e. $(\partial/\partial \bar{z}) K_2(h, D) = h$ holds on D.

Because of these properties, a substantial portion of the properties of holomorphic functions of a single variable may be obtained as relatively simple consequences of (2).

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[1] This result was discovered by A. B. Aleksandrov [Mat. Sb. **118(160)** (1982), 147–163 (in Russian)] and independently by E. Løw [Invent. Math. **67** (1982), 223–229] using a construction of Hakim and Sibony [Invent. Math. **67** (1982), 213–222].

There is a large literature concerned with obtaining generalizations of (3) to various domains in $\mathbf{C}^n$. These have been used to obtain deep and precise results, such as the uniform estimates for the solution of $\bar{\partial}$. The kernels involved give new and interesting classes of singular integral operators. Also there are relations between some of these operators and the Bergman and Szegö projections. Despite this progress, however, the construction of these kernels remains rather difficult, and it has not been possible to find a single formula with the properties (a), (b) and (b′) which make (2) so powerful.

5. Nonexistent Riemann mapping theorem. It is an old observation that the ball $\mathbf{B}^2 = \{|z_1|^2 + |z_2|^2 < 1\}$ is not biholomorphically equivalent to the bidisk $\Delta^2 = \{\max |z_1|, |z_2| < 1\}$; thus there is no Riemann mapping theorem for domains in $\mathbf{C}^n$ when $n \geqslant 2$. In the case of $\mathbf{B}^2$ and Δ^2, the nonexistence of a mapping may be explained on the basis that $\partial\mathbf{B}^2$ is strongly pseudoconvex while the boundary surfaces $\partial\Delta \times \Delta$ and $\Delta \times \partial\Delta$ of $\partial(\Delta^2)$ are "flat". Yet there are still boundary obstructions to the existence of a mapping between hypersurfaces S_1 and S_2 even if the surfaces are both strongly pseudoconvex or both Levi flat. In the strongly pseudoconvex case there are even an infinite number of biholomorphic invariants at each point $p \in S = \partial\Omega$, each one depending on a finite number of derivatives of $\partial\Omega$ at p. Because of these boundary invariants it is quite "unlikely" that any two strongly pseudoconvex domains Ω_1 and Ω_2 are biholomorphically equivalent. On the other hand, the invariants are sufficiently difficult to compute that it is essentially impossible to decide the equivalence of Ω_1 and Ω_2 on the basis of them.

Implicit in our discussion of obstructions to mappings in terms of boundary invariants is the regularity theorem: *A biholomorphism f: $\Omega_1 \to \Omega_2$ between smoothly bounded strongly pseudoconvex bounded domains extends smoothly to* $\bar{\Omega}_1$. This result was first proved by C. Fefferman, but by now it has been greatly generalized, and many new proofs have been given.

In the same vein of nonexistence, one could ask, given a domain $\Omega \Subset \mathbf{C}^n$, whether $\mathrm{Aut}(\Omega) = \{\text{biholomorphisms } f\colon \Omega \to \Omega\}$ contains any elements other than the identity? In view of the local invariants, one expects the existence of automorphisms only for special Ω. There is also the following *Compactness Theorem*: *If Ω is strongly pseudoconvex, then* $\mathrm{Aut}(\Omega)$ *is compact unless* $\Omega \approx \mathbf{B}^n$.

6. Conclusion. Krantz's method of covering the material of §§3, 4 and 5 is not to give comprehensive treatises but to present selected theorems and highlights, serving as introductions to these currently active areas of research. The treatment of boundary values is given from the point of view of harmonic analysis. This follows the philosophy of E. Stein and his collaborators and uses the techniques of maximal functions and coverings by "balls" whose shapes are adapted to the geometry of the boundary. Holomorphic mappings are discussed geometrically. The main results are the Compactness and Regularity Theorems stated in §5. The biholomorphic inequivalence of $\mathbf{B}^n$ and Δ^n is a recurrent theme. Since Krantz did not research his historical remarks very thoroughly, we might point out the old result of K. Stein and H. Rischel that there is no proper correspondence [more general than a proper mapping] between Δ^n and $\mathbf{B}^n$.

Integral formulas are difficult to discuss in just one chapter. For one thing, there are many different formulas; for another, their construction and use has remained a rather technical enterprise. Krantz limits himself to constructing and estimating the "Leray-Stokes" formula of Henkin in the case $n = 2$, a choice which strikes a good compromise between generality and technicality.

As a text, this book should be excellent for a second course on complex analysis. It covers many of the basic results and connects them up with harmonic analysis and P.D.E.; and the final three chapters provide an introduction to more specialized material.

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Hardy spaces on homogeneous groups, by G. B. Folland and E. M. Stein, Mathematical Notes, Princeton Univ. Press, Princeton, N. J., 1982, 284 pp., $13.00. ISBN 0-6910-8310-X

"The original and basic concept of functional analysis is that of an operator. Just as an analytic function has its complete domain of definition, so an operator has a complete set of spaces on which it can be examined." This maxim is from the 1966 survey article of S. G. Krein and Yu. I. Petunin on interpolation spaces [**5**], who also refer to "the victory of the operators over the spaces." For a nice example of this phenomenon, consider some recent results of Marshall, Strauss, and Wainger [**6**] concerning the nonlinear Klein-Gordon equation

$$\text{(NLKG)} \qquad v_{tt} - \Delta v + v + \lambda |v|^{\alpha} v = 0$$

and its linearization at $v = 0$,

$$\text{(KG)} \qquad u_{tt} - \Delta u + u = 0.$$

Here Δ is the Laplace operator on $\mathbf{R}^n$, and one assumes $\alpha > 0$, $\lambda > 0$. Given a function f in $L^2(\mathbf{R}^n)$, there is a unique (weak) solution $u(x, t)$ to (KG) with initial data $u(x, 0) = 0$ and $u_t(x, 0) = f(x)$. Let $T_t\colon f \to u(\cdot, t)$ be the operator which takes initial velocity f into position $u(\cdot, t)$ at time t. The problem is to construct a finite energy solution to (NLKG) which is asymptotic in the energy norm as $t \to -\infty$ to $T_t f$. As was shown previously by Strauss, this can be reduced to the problem of obtaining certain bounds (in terms of t) for the norm of the linear operator T_t from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, with $1/p + 1/q = 1$. This is carried out in [**6**] by using results of Fefferman and Stein about the operator $(1 - \Delta)^{is}$, $s \in \mathbf{R}$, on the space BMO, together with the Stein interpolation theorem applied to a holomorphic family of operators T_t^{α} $(\alpha \in \mathbf{C})$ containing T_t. The result is that the nonlinear scattering problem at $t = -\infty$ has a solution when the exponent α in (NLKG) and the space dimensionality n satisfy $4/n \leqslant \alpha \leqslant [4/(n - 1)]$. In particular, the physically interesting case $n = 3$, $\alpha = 2$ is included in this treatment.