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Non-relativistic quantum dynamics, by W. O. Amrein, Mathematical Physics Studies, vol. 2, Reidel, Dordrecht, Holland; Boston, U. S. A.; London, England, 1981, viii + 237 pp., paper Dfl. 70.00/US $34.50. ISBN 9-0277-1324-3

The author is one of the foremost expositors (see, for example, the more comprehensive book Amrein, Jauch and Sinha [**1**]) of the scattering theory of Schrödinger operators and the present introductory book is a welcome addition. I use the term introductory here purposely but with two qualifications.

First, as the author states, the book is in the spirit of an essay on the so-called time-dependent methods as viable alternatives to the more ponderous time-independent methods. Moreover, after some perusal, one sees that the main thrust of the book is to present a treatment of some of the very most recent research results in the time-dependent potential scattering theory. These go under the name "geometric" methods. They have emerged only recently in the pioneering work of V. Enss [**2**] (see also Enss [**3, 4**]), and have been advanced by Simon [5], Mourre [6], Perry [7], Yafaev [8], Davies [9], Kitada and Yajima [10], Muthuramalingam and Sinha [11], and others.

The first qualification, therefore, is that this book, although introductory in spirit, deals with an advanced subject of high recent research activity.

The second qualification concerns the level of the book if considered as a textbook. The author states, and I clarified this in a query to him, that the book has been used successfully in its full form for an undergraduate course in mathematical physics. In roughly 36 hours of lecture, plus some exercise sessions, the first five chapters can be covered. These are (1) Linear operators in Hilbert space, (2) Self-adjoint operators, Schrödinger operators, (3) Hilbert-Schmidt and compact operators, (4) Evolution groups, (5) Asymptotic properties of evolution groups. The last chapter is, by the way, (6) Scattering theory, and deals mainly with results obtained during the last three years concerning rigorous bounds on scattering cross-sections.

The second qualification that I would like to make is that upper-division undergraduates in Europe are at about the same age and mathematical level as second year graduate students in the United States, and that as a textbook we would be more likely to consider it in a typically second year graduate course in functional analysis or mathematical physics or else perhaps afterwards for a special topics or seminar type course.

The advent of the recent time-dependent "geometric" methods was deemed of sufficient importance to cause a delay of the third volume of Reed and Simon [12] to a publication date later than that for the fourth volume, in order to include the geometric methods and results in the former. Thus one may find a brief presentation of them therein [12, vol. III, pp. 331–344]. One has also the survey by Enss [4]. But the book under review is the only essentially self-contained treatment presently known to the reviewer.

The "geometric" methods are indeed geometric, as may be illustrated as follows. A vector f in the Hilbert space $\mathfrak{L}^2(R^n)$ is called a scattering state for a given evolution group U_t at $t = +\infty$ if the limit as $t \to +\infty$ of $\|\chi_r U_t f\|^2$ is zero for every $r > 0$, where χ_r is the characteristic function of the ball in R^n of radius r. Recall that in the Hilbert space formulations of quantum mechanics the quantity $\|\chi_r U_t f\|^2$ may be interpreted as the probability of finding the evolving state $U_t f$ in the r-ball at time t. Thus a scattering state is one which eventually escapes every finite ball.

The "geometric" methods that are the central issue in this book are concerned with the more specific and difficult question of asymptotic completeness of a quantum mechanical scattering system $\{H_0, H = H_0 + V\}$. Here one may for simplicity think of H_0 as the Laplacian Δ and the potential V as multiplication by a function which falls off rapidly at ∞. Given a "bare" Hamiltonian H_0 and its evolving states $U_t^0 f \equiv e^{-iH_0 t} f$ and the full Hamiltonian H and its evolving states $U_t f \equiv e^{-iHt} f$, how may we deduce properties about the scattering states of H from those of H_0? This is a recurrent and principal underlying theme for many of the questions which arise in scattering theory.

A key idea in the geometric methods used for asymptotic completeness is the following. Let a state f evolve under U_t^0 and imagine it to be initially localized at $t = 0$ in a small r-ball. We may ignore the probabilistic considerations and think of it as composed of particles at points x in the r-ball. Under the group U_t^0 a particle is translated to $x + vt$, where v = velocity = the momentum p if we for simplicity normalize mass to one. Then, because

$$|x + vt|^2 = |x|^2 + |vt|^2 + 2tx \cdot v,$$

if we require that $x \cdot v \geqslant 0$ we may conclude that the particle moves away. If we also impose a minimum positive velocity for all of the particles of the state f, then the support of f eventually escapes every finite ball as $t \to \infty$.

This means that if the state f is in the so-called positive momentum subspace $\mathcal{D}_+ = D_+ \mathcal{H}$ (see below) and localized near the origin, then $U_t^0 f$ is localized far away for large times. One then needs to bring in the potential V, using its falloff properties, to try to deduce a similar statement about the full evolution $U_t f$.

Let us keep in mind the above ideas and see how the author proceeds.

Chapters 1, 2, and 3 are preliminary, as is Chapter 4 except for one key move. In §4.2 the functional calculus for the infinitesimal generator A of a unitary evolution $U_t \equiv e^{-iAt}$ is developed in a time-dependent way. Of course the spectral family E_λ and resolvent R_λ of A creep in from time to time in the following material, but the expression

$$\phi(A) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} \tilde{\phi}(t) U_t^* \, dt \tag{4.37}$$

for the function $\phi(A)$ in terms of U_{-t} and the Fourier transform $\tilde{\phi}$ of ϕ tells you already where you may expect to go. The mean ergodic theorem (4.71) begins the linkage to the geometric theory.

Chapter 5 is the main one and begins with the definitions of scattering states and scattering states on the time average (5.7), and establishes some of their properties. On page 133 one finds just a hint that the principal goal of the chapter, indeed in my view of the book, is Proposition 5.34 on page 176, the asymptotic completeness theorem for short range potentials. Note that these potentials were clearly described on pages 53, 64, 163, the terminology short range meaning that they fall off more rapidly then the Coulomb potential. Note also that they are multiplication operators; we will return to this point later.

The asymptotic completeness theorem states that the wave operators $\Omega_\pm =$ $s\text{-}\lim U_t^* U_t^0$ as $t \to \pm\infty$ exist and that

$$\mathcal{R}(\Omega_\pm) = \mathcal{H}_p(H)^\perp = \mathcal{H}_{ac}(H) = \mathfrak{M}_\infty^\pm(H). \tag{5.95}$$

That is, the ranges $\mathcal{R}(\Omega_\pm)$ of the wave operators coincide and are comprised exactly of the scattering states $\mathfrak{M}_\infty^\pm(H)$ of the full Hamiltonian H. Furthermore the essential spectrum $\sigma_e(H) = [0, \infty)$ and is absolutely continuous except possibly for some embedded positive eigenvalues.

Continuing with the technical development, on page 153 the abstract conditions (C1) through (C8) are given. The condition (C1) and the operators D_+ and D_- are important and will lead to a "geometric splitting of the velocities". On page 166 the classical version of this splitting is explained and on page 167 the connection to the generator $B = \frac{1}{2}(P \cdot Q + Q \cdot P)$ of the dilation group U_θ is given. Here the author prefers to B the version A_0 due to Yafaev **[8]** which has the advantage (page 170) that one obtains a simultaneous diagonalization of the bare Hamiltonian H_0 and A_0, the latter to a one-dimensional momentum operator:

$$U_0 A_0 U_0^{-1} = 2i \frac{\partial}{\partial \lambda}. \tag{5.85}$$

As a matter of experience and in retrospect, from such a "regular representation" one can expect many "outgoing" and "incoming" conclusions to follow, and from (5.85) it is only five more pages to the main result Proposition 5.34.

We hope that this description is sufficient for those who might wish to use the book. The reader should be aware of the highly mathematical nature of this book. As such, it will certainly be too mathematically technical for all but the most mathematical of physicists. On the other hand it will certainly be too physically limited for those wishing a more physically technical treatment such as that of Newton **[13]**.

We would like to close this review with a few observations.

A minor bibliographical point: the reference **[GR]** ended up with a different title, see **[14]** here.

A minor quibble: the inference on page 112 that

$$H_{ac}(A) = \mathrm{sp}\left\{ f \,\middle|\, \int \left| \langle f, U_t f \rangle \right|^2 dt < \infty \right\}$$

need be found in **[1]** of **[VW]** should be corrected. This was shown in **[15]** in the time-independent framework, see **[16**, Lemma 2 and Remark 2] for the time-dependent version.

Enss **[2, 3]** originally used a rather complicated phase space decomposition. Mourre **[6]** simplified the arguments by use of a partition of unity of the type $D_+ + D_- = I$ in terms of "incoming" and "outgoing" subspaces $\mathfrak{D}_\pm$. This is reminiscent of such subspaces in the accoustic scattering theory of Lax and Phillips **[17]**, and it is difficult to say with certainty where this all started, for it goes all the way back to the canonical commutation relations.

The generators B, A_0, and others now appearing in the applications of the geometric and dilation group methods possess interesting commutation properties with the bare Hamiltonian H_0. For example, the dilation group $(U(\theta)f)x = e^{3\theta/2} f(e^\theta x)$ in $\mathfrak{L}^2(R^3)$ with generator B given above on the space $\mathfrak{S}$ of rapidly decreasing functions satisfies $i[H_0, B] = 2H_0$. Such relations are useful not only for asymptotic completeness but also for related considerations such as time delay; see for example **[18]**.

Although the potentials V need not necessarily be restricted to multiplication operators, usually, and in the book under review as we observed earlier, they are. Note also that once one obtains a representation such as (5.85), or more precisely, its transform, one is working with multiplication operators, e.g., on coordinate spaces. Therefore, given potentials V that are multiplication operators, any obtained relation involving terms $[H_0, A]$ where A is an operator that has been represented as a multiplication operator is preserved with terms $[H, A]$. We may call this a general "commutator invariance principle", and we would like to assert that it underlies in particular much of the success of the dilation geometric methods.

Finally, the "bottom line" in this book is indeed its last line, and we quote: "What is still missing is a time-dependent proof of asymptotic completeness in the general case." Such results are now being obtained and will be reported on, for example, by Enss **[19]**.

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