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NOWHERE SOLVABLE HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS[1]

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Nirenberg showed in [4] that the Lewy operator $L_0 = \partial/\partial x + i\partial/\partial y - 2i(x+iy)\partial/\partial u$ may be perturbed to obtain an operator L_1 such that if $L_1 h = 0$ in a connected neighborhood of the origin then h is a constant in the neighborhood. In this note we eliminate the restriction to neighborhoods of a distinguished point and show that such operators are dense in the natural topology. We then announce a similar result for certain systems.

We consider operators defined on some open set $\Omega \subset \mathbf{R}^3$ where perhaps Ω is all of $\mathbf{R}^3$. Let $S = \{L = \sum_{j=1}^3 \alpha_j(x)\partial/\partial x_j;\ \alpha_j \in \mathbf{C}^\infty(\Omega, \mathbf{C})\}$. As the topology on S we take the one induced by the usual topology on $\mathbf{C}^\infty(\Omega, \mathbf{C})$, namely uniform convergence of all derivatives on compact subsets. Note that this topology is metrizable and thus any countable intersection of open dense sets is dense.

DEFINITION. An operator $L \in S$ is aberrant if every C^λ function h, with $\lambda > 1$, satisfying $Lh = 0$ on some open connected subset of Ω is constant on that subset.

THEOREM. *The aberrant operators are dense in S.*

REMARKS. 1. The proof which follows is surprisingly simple and avoids the complicated construction in [4]. However Nirenberg's result only required that $h \in C^1$.

2. In our previous paper [1] we showed that any operator in S with L, $\bar{L}$, and $[L, \bar{L}]$ linearly independent may be perturbed to an operator L_1 such that if $L_1 h = 0$ near the origin then $dh = 0$ at the origin. As we shall see, a simple Baire category argument allows us both to conclude that h is a constant and to eliminate the distinguished point.

3. The denseness holds in a finer topology since we work with only compactly supported perturbations. In particular we may find an aberrant operator L for which L, $\bar{L}$, and $[L, \bar{L}]$ are linearly independent at each point of $\mathbf{R}^3$. Thus there exists a strictly pseudo-convex CR structure on $\mathbf{R}^3$ with only the constants as local CR functions.

We start the proof by defining a subspace of S. For each $P \in \Omega$ let $S_P = \{L \in S;\ \text{if } v \in C^1(\Omega) \text{ and } Lv = 0 \text{ in a neighborhood of } P, \text{ then } dv(P) = 0\}$.
Here $dv(P)$ represents the differential of v acting on $T_P\mathbf{R}^3$.

LEMMA. *For each $P \in \Omega$, S_P is dense in S.*

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PROOF. Since this is an approximation result we may assume that L is real analytic in some neighborhood of P and that L, $\bar{L}$ and $[L,\bar{L}]$ are linearly independent there. We now follow the procedure used in [**1**]. Since L is real analytic there exist solutions $Lh_1 = 0$ and $Lh_2 = 0$ with $dh_1 \wedge dh_2 \neq 0$ near P. If $F(z,w)$ is a holomorphic function then $h(x) = F(h_1(x), h_2(x))$ also satisfies $Lh = 0$. Using this observation plus the fact that L, $\bar{L}$ and $[L,\bar{L}]$ are linearly independent we may introduce coordinates (x,y,u) on Ω near P such that $z = x+iy$ and $w = u + iG(x,y,u)$ are solutions of $Lh = 0$ and G has the form $G = |z|^2 + \alpha z u + \bar{\alpha}\bar{z}u + bu^2 + \mathcal{O}((|z|+|u|)^3)$. In these coordinates $L = \phi(x)((1+iG_u)\partial/\partial\bar{z} - iG_{\bar{z}}\partial/\partial u)$ where $\phi(P) \neq 0$.

We consider the operator R defined by $L = \phi R$. It suffices to find some R_1, defined in a neighborhood of the origin and uniformly close, along with its derivatives up to some large order, to R in that neighborhood, with the property that if v is C^1 and $R_1 v = 0$ near the origin, then $dv(0) = 0$. For then we transform back to the original coordinates and extend R_1 appropriately to obtain our operator L_1.

Let $\Gamma(\lambda) = \{(z,u); w(z,\bar{z},u) = \lambda\}$. Since $w = u + iG(x,y,u)$, with G having the form given above, one can find some curve $\operatorname{Im}\lambda = \nu(\operatorname{Re}\lambda)$ passing through the origin in the λ-plane such that $\Gamma(\lambda)$ is a simple closed curve for $\operatorname{Im}\lambda > \nu(\operatorname{Re}\lambda)$, a point for $\operatorname{Im}\lambda = \nu(\operatorname{Re}\lambda)$, and empty for $\operatorname{Im}\lambda < \nu(\operatorname{Re}\lambda)$. Let $\tilde{\Omega}$ be a neighborhood of the origin symmetric with respect to the curves $\Gamma(\lambda)$ in the sense that whenever $\Gamma(\lambda) \cap \tilde{\Omega}$ is nonempty, then $\Gamma(\lambda) \subset \tilde{\Omega}$. Next let $\{C_i\}$ be a sequence of pairwise disjoint closed discs in the λ-plane, lying above the curve $\operatorname{Im}\lambda = \nu(\operatorname{Re}\lambda)$ and collapsing down to the origin. Let $T_i = \{(z,u); w(z,\bar{z},u) \in C_i\}$ be the corresponding closed topological solid torus. Now consider a function h with $Rh = 0$ in $\tilde{\Omega} - \bigcup T_i$. One may show that $\int_{\Gamma(\lambda)} h\,dz$ is a holomorphic function of λ as long as $\Gamma(\lambda) \subset \tilde{\Omega} - \bigcup T_i$. This holomorphic function becomes zero on the curve $\operatorname{Im}\lambda = \nu(\operatorname{Re}\lambda)$ and so is identically zero. It follows that $\int\int_{\partial T_i} h\,dz \wedge dw = 0$. But for any set with a smooth boundary

$$\int\int_{\partial T} h\,dz \wedge dw = \int\int\int_T dh \wedge dz \wedge dw = 2i \int\int\int_T Rh\,dx\,dy\,du.$$

Now let $f_1(z,\bar{z},u)$ be a C^∞ function which is positive in each T_{2i+1} and zero everywhere else; similarly for f_2 and T_{2i}. If $Rh = f_1 h_u + f_2 h_z$ then

$$\int\int\int_{T_{2i+1}} f_1 h_u\,dx\,dy\,du = 0 \quad \text{and} \quad \int\int\int_{T_{2i}} f_2 h_z\,dx\,dy\,du = 0.$$

It follows that $h_u(0) = h_z(0) = 0$. Since $Rh = 0$ at the origin, one also has that $h_{\bar{z}}(0) = 0$. Thus we take $R_1 = R - f_1\partial/\partial u - f_2\partial/\partial z$ and use it, in the manner given above, to establish the lemma.

Our Baire category argument is a modification of that of Lewy [**3**]. Let $\{P_j\}$ be a countable dense set of points in Ω, N_j the open ball of radius j^{-1} centered at P_j, $\|h\|_{\lambda,j}$ the Hölder norm for $C^\lambda(N_j)$ and $|\omega(P_j)|$ any norm on the three dimensional vector space of co-vectors at P_j.

We let $E_{j,m,n}$ be the set of all those vector fields $L \in S$ for which there exists a function $h \in C^{1+1/n}(N_j)$ such that

1. $Lh = 0$ in N_j,
2. $||h||_{1+1/n,j} \leq m$,
3. $|dh(P_j)| \geq 1/m$.

If $L \in$ closure $(E_{j,m,n})$ then there exists some $h \in C^1(N_j)$ such that $Lh = 0$ in N_j and $|dh(P_j)| \geq 1/m$. (In fact more is true: $h \in C^{1+1/n}$ and so $E_{j,m,n}$ is closed.) Thus $L \notin S_{P_j}$. Now the fact that S_{P_j} is dense implies that $E_{j,m,n}$ is nowhere dense. By the Baire theorem there exists a dense set of vector fields none of which belongs to any $E_{j,m,n}$. Any one of these vector fields has the property asserted in the theorem.

To present a result for higher dimensions we consider complex vector fields $L_1, \ldots, L_n$ on $\Omega \subset \mathbf{R}^{2n+1}$ with the following properties:

1. $L_1, \ldots, L_n, \bar{L}_1, \ldots, \bar{L}_n$ are linearly independent.
2. $[L_j, L_k] = 0 \bmod\{L_1, \ldots, L_n\}$.
3. The Levi form has $n-1$ eigenvalues of one sign and one of the opposite sign.

Such vector fields form a space τ which we topologize using $C^\infty(\Omega, C)$. (It is of no concern that τ is not a linear space.)

It is a well-known result of Hörmander that under condition 3 (for $n > 1$) any distribution which satisfies the system $L_j h = 0$ is necessarily C^∞. So we now say that the system of operators $L_1, \ldots, L_n$ is *aberrant* if every distribution solution to $L_j h = 0$ for $j = 1, \ldots, n$ on an open, connected subset of Ω is a constant function on that subset.

THEOREM. *The aberrant operators are dense in τ.*

See [**2**] for the proof. Here we only note that the perturbation process in [**1**] for $n > 1$ did not yield $dh = 0$ and so a slightly more complicated procedure must be used. The difficulty, of course, is that one must only work with perturbations which preserve condition 2 (formal integrability).

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