ON FIXED POINTS OF AUTOMORPHISMS
OF FINITELY GENERATED FREE GROUPS

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ABSTRACT. If ϕ is an automorphism of a finitely generated free group,
then $\mathrm{Fix}(\phi)$ is finitely generated.

0. Introduction. A conjecture attributed to G. P. Scott [**St2**] states that
the fixed points of an automorphism ϕ of a finitely generated (fg) free group F
is fg. It is known that this is the case if ϕ is periodic [**D-S**] or geometric [**J-S**],
where ϕ is called geometric if it is the induced map on π_1 of a homeomorphism
of (M, x), where M is a compact bounded two-dimensional manifold and $x \in
M$. In a recent article [**G2**] we showed that Scott's conjecture is valid for
automorphisms of π_1 of a finite graph induced by change of maximal trees.

In this article we announce the affirmative solution of Scott's conjecture.
The methods involved are an extension of methods of [**G2**] and blend com-
binatorics, topology, and group theory.

1. Graphs. We work with a combinatorial notion of a graph introduced
in [**G1**] (and based on an idea of Serre's [**S**]). A graph X is a nonempty set
with involution $x \to \bar{x}$ (so $\bar{\bar{x}} = x$) and retraction $i: X \to V(X)$, where $V(X)$
is the fixed point set of $x \to \bar{x}$. Thus $i(i(x)) = i(x)$. One defines $\tau(x) =: i(\bar{x})$.
Intuitively, $V(X)$ is the set of vertices, $X - V(X) =: E(X)$ is the set of edges,
i is the initial vertex and τ is the terminal vertex.

A morphism $f: X \to X'$ of graphs is a function such that $f(\bar{x}) = \overline{f(x)}$,
and $i(f(x)) = f(i(x))$, $x \in X$. The category of graphs has a final object, fibre
products and push outs. In addition, there is a geometrical realization functor
BX which assigns to X a 1-dimensional CW complex. This permits us to use
the geometrical language of maximal trees, fundamental group and homotopy
equivalence, although the notions have purely combinatorial definitions in the
category of graphs.

A morphism $f: X \to Y$ is called an *immersion* [**St1**] if $f_v: \mathrm{Star}_v(X) \to
\mathrm{Star}_{f(v)}(Y)$ is injective for each $v \in V(X)$. Here $\mathrm{Star}_v(X) = \{e \in E(X) \mid ie =
v\}$. Immersions preserve reduced paths and hence induce injective maps on
π_1 [**St1**]. Of crucial importance in our work is the *degenerate set* Df of a
morphism $f: X \to Y$; here $Df =: \{x \in X \mid f(x) \in V(Y)\}$.

If $v \in V(X)$ and $f: X \to Y$ is a morphism, denote by f_* the induced map
$\pi_1(X, v) \to \pi_1(Y, f(v))$.

1.1. THEOREM. *If $\phi \in \mathrm{Aut}(\pi_1(Y))$ where Y is a finite graph with one vertex, there exists a finite graph X, $v \in V(X)$, and morphisms f, $f' : X \to Y$ s.t.*

(1) f_* *and* f'_* *are surjective with the same kernel,*

(2) $f'_* \circ f_*^{-1} = \phi$, *and*

(3) Df *and* Df' *contain maximal trees of* X.

The proof makes use of Whitehead's algorithm [**L-S**] for decomposing ϕ as a product of Whitehead automorphisms, and a theorem of ours, [**G2**, 5.5], representing a Whitehead automorphism as a change-of-maximal-tree automorphism (blow-up followed by blow-down in the aeolean terminology of [**G1**]).

After modifying X, f, f' in 1.1 by edge collapses, edge folds [**St1**] and by removing extraneous edges, one proves

1.2. THEOREM. *In the notation of* 1.1, *one may assume* f, f' *satisfy*

G1. Df, Df' *are maximal trees of* X,

G2. $f \mid (X - E(Df))$ *is an immersion and* $f' \mid (X - E(Df'))$ *is an immersion,* and

G3. f_* *and* f'_* *are surjective with the same kernel.*

In addition, G1, G2, and G3 imply

G4 (*"path surgery"*). *If* x, $y \in E(X) - E(Df)$ *are such that* $f(x) = f(y)$, then

$$l(\bar{x} \cdot [ix, iy]_{Df} \cdot y) > l([\tau x, \tau y]_{Df});$$

and similarly for f' replacing f. Here $[v, w]_{Df}$ is the geodesic in the maximal tree Df from vertex v to vertex w, and "l" denotes the length of a path.

Let $\mathcal{E}_v = \{u \in \pi_1(X, v) \mid f_*(u) = f'_*(u)\}$. It is easily seen that $f_*(\mathcal{E}_v) \cong \mathrm{Fix}(\phi) = \{\eta \in \pi_1(Y) \mid \phi(\eta) = \eta\}$. With these preliminaries we can state our main result.

1.3. THEOREM. *Assume* $\phi \in \mathrm{Aut}(\pi_1(Y))$, *where* Y *is a finite graph with one vertex. Assume* f, $f' : X \to Y$, $v \in V(X)$ *satisfy* G1–G4 *in* 1.2, *with* $f'_* \circ f_*^{-1} = \phi$. *Then* $f_*(\mathcal{E}_v) \cong \mathrm{Fix}(\phi)$ *is finitely generated.*

We remark that in all examples we have studied, as well as in the results of [**D-S**], [**J-S**], and [**G2**], $\mathrm{rank}(\mathrm{Fix}\,\phi) \leq h_1(Y)$ (in the situation of [**G2**] $X \simeq Y$). We do not know if this inequality holds in general.

We shall outline the proof of 1.3 in the remainder of this article. Properties G1–G4 of f, $f' : X \to Y$ will be assumed throughout without further mention.

2. Representation theorems.

2.1. A reduced path $(x_1, \ldots, x_n)$ in X [**St1**] is called f *reduced* if, after deleting all $x_i \in Df$ and applying f to the remaining edges, the resulting path in Y is reduced. The path p in X is called (f, f') *reduced* if it is both f- and f' reduced. A path p in X is called *invariant* if $f_*p = f'_*p$, where f_*p denotes the reduced path in Y in the homotopy class of $f(p)$. An (f, f') reduced invariant path p in X is called *minimal* if no nontrivial initial segment of p is invariant. We denote by $\mathcal{M}$ the set of minimal (f, f') reduced invariant paths.

2.2. THEOREM. *If* $\eta \in f_*(\mathcal{E}_v)$, *then there are only finitely many* (f, f') *reduced paths* w *in* X *with* $f_*w = f'_*w = \eta$; *and they are determined by a simple algorithm.*

This algorithm is based on the algorithm of [G2, §4].

2.3. THEOREM. *For each path w in X $\exists$ an (f, f') reduced path w_1 s.t.*
(1) *$iw = iw_1$, $\tau w = \tau w_1$, and*
(2) *$f_* w = f_* w_1$, $f'_* w = f'_* w_1$.*

The proof is accomplished by doing path-surgeries. The process converges by G4.

2.4. COROLLARY. *If $\eta \in f_*(\mathcal{E}_v)$, $\exists$ an (f, f') reduced circuit w based at v s.t. w is invariant and $f_* w = \eta$.*

Writing w in 2.4 as a path product of paths in $\mathcal{M}$, one deduces

2.5. COROLLARY. *$f_*(\mathcal{E}_v) = f_*(\langle \mathcal{M} \rangle \cap \pi_1(X, v))$.*

Here X is immersed in $Z =: X/[V(X) = pt]$ and the intersection on the right-hand side takes place in $\pi_1(Z)$.

3. Uniqueness theorems.

3.1. DEFINITION. If $e \in E(X)$, let $\mathcal{M}_e$ denote the set of paths $p \in \mathcal{M}$ with initial edge e. A path of minimal length in $\mathcal{M}_e$ is called *standard*. If p and p' are paths in X with $ip = ip'$, $\tau p = \tau p'$, $f_*(p) = f_*(p')$ and (of necessity, by G3) $f'_*(p) = f'_*(p')$, we write $p \overset{\circ}{=} p'$.

3.2. THEOREM. *If p and p' are in $\mathcal{M}_e$ and if p is standard, then $f_*(p)$ is an initial segment of $f_*(p')$.*

The proof is accomplished by careful examination of an algorithm which constructs all elements of $\mathcal{M}$.

3.3. COROLLARY. *If p and p' are in $\mathcal{M}_e$ and are both standard, then $p \overset{\circ}{=} p'$.*

4. Main result.

4.1. LEMMA. *Let w, w' be invariant paths in X such that $i(w) = i(w')$ and $f_*(w) = f_*(w')$. Then $\tau(w) = \tau(w')$.*

We now reduce the (possibly infinite) set $\mathcal{M}$. For each $e \in E(X)$ such that $\mathcal{M}_e \neq \varnothing$, pick one standard path w_e. Let $\mathcal{M}'$ be the set of w_e chosen.

4.2. THEOREM. *$f_*(\langle \mathcal{M}' \rangle \cap \pi_1(X, v)) = f_*(\langle \mathcal{M} \rangle \cap \pi_1(X, v))$.*

One shows by induction on $l(f_*(w))$ that if $w \in \mathcal{M}$, then $w \overset{\circ}{=} w'$, where w' is a path product of paths in $\mathcal{M}'$.

PROOF OF THEOREM 1.3. By 2.5 and 4.2 it suffices to prove that $\langle \mathcal{M}' \rangle \cap \pi_1(X, v)$ is finitely generated. But $\mathcal{M}'$ is a finite set, so the result follows from a theorem of Howson's [St1, 5.6].

One can deduce from H. Neumann's inequality, [G1, 4.1], the inequality rank $(\mathrm{Fix}(\phi)) - 1 \leq 2(\#E(X) - 1) \cdot (h_1(X) - 1)$, where $h_1(X)$ is the first betti number of X, provided $\mathrm{Fix}(\phi) \neq \{1\}$.

5. Automorphisms with few fixed points.

5.1. THEOREM. (*"Antisymmetry principle"*). *Suppose $w \in \mathcal{M}$ has initial edge in Df. Then w must have its terminal edge in Df'.*

5.2. PROPOSITION. *Suppose $e \in E(Df)$ is s.t. if $f'(e) = f(x)$ for any $x \in E(X)$, then $[\tau e, ix]_{Df}$ has initial edge $\bar{e}$. Then $\mathcal{M}_e = \varnothing$.*

From 5.1 and 5.2, together with 2.5, we can produce examples of automorphisms with no nontrivial fixed points. In particular all the examples of [Sq] can be treated by our methods. This is of interest in accumulating evidence for Stallings' conjecture [St2], that a PV automorphism ϕ of a *fg* free group of rank ≥ 3 has $\mathrm{Fix}(\phi) = \{1\}$.

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