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PHYSICAL SPACE-TIME AND NONREALIZABLE CR-STRUCTURES

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Abstract. Space-time views leading up to Einstein's general relativity are described in relation to some of Poincaré's early ideas on the subject. The basic geometry of twistor theory is introduced as it arises both from Minkowski space-time and the more general curved Einstein models. It is shown how this provides a CR-structure (this being, in essence, another of Poincaré's pioneering concepts) in a natural way. Nonrealizable CR-structures can arise, and an example is presented, due to C. D. Hill, G. A. J. Sparling and the author, of a complex manifold-with-boundary which cannot be extended as a complex manifold beyond its C^∞ boundary.

1. Introductory remarks. The nature of physical geometry was something that held considerable interest for Poincaré, and he often referred to it in his more popular writings. Moreover, it was Poincaré who first clearly understood the physical transformation group of special relativity—arising as a group of symmetries of Maxwell theory—and he suggested that the relativity principle concerned might hold for physics generally. This dates back to 1899, six years before Einstein's first paper on relativity (cf. Poincaré (1906), (1954) for details). It is fitting, therefore, that this symmetry group should be now very commonly referred to as the Poincaré group (otherwise known as the inhomogeneous Lorentz group, where simply "Lorentz group" now normally refers only to the related homogeneous group).

Poincaré also had interesting and, to some extent, insightful things to say about the possibility that physical space might have a non-Euclidean geometry. But here his instincts appear ultimately to have let him down. For in "Science and Hypothesis" Poincaré (1905) wrote

> "What then, are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the

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> metric system is true, and if the old weights and measures are false; if Cartesian coordinates are true and polar coordinates false. One geometry cannot be more true than another; it can only be more convenient. Now, Euclidean geometry is, and will remain, the most convenient: 1st, because it is the simplest, and it is not so only because of our mental habits or because of the kind of direct intuition that we have of Euclidean space; it is the simplest in itself, just as a polynomial of the first degree is simpler than a polynomial of the second degree; 2nd, because it sufficiently agrees with the properties of natural solids, those bodies which we can compare and measure by means of our senses."

It is, of course, no discredit to Poincaré that he should have had no inkling of that extraordinary development, which emerged three years after his death, namely Einstein's *general* theory of relativity. Nevertheless, I find it rather remarkable that Poincaré, with his deep geometric and philosophical insights, should apparently have had such a rigid feeling that geometries other than Euclidean could have no chance of providing accurate and useful descriptions of physical space according to some future theory. His remarks on the conventionality of geometry and on the fact that the geometry we use in physics is of necessity an idealization are, indeed, quite profound and insightful. But he seems to have made a rather bad mistake in his reasoning, his critical faculties being perhaps dulled by an erroneous (but natural enough) intuitive presumption that in "our world" the geometry actually *is* Euclidean!

According to modern cosmology, it is quite on the cards that the large-scale spatial geometry of the universe may indeed be closely in accord with Lobachevskian geometry—that geometry which held such a fascination for Poincaré the mathematician, yet which had been rejected as inevitably physically inappropriate by Poincaré the philosopher! It would be interesting to know how Poincaré would have reacted to present-day cosmology. Perhaps there is an object lesson here for all of us.

Even if observation finally turns against its present slight preference for the Lobacheveskian spatial geometry—and also tells against models with a positively curved spatial geometry—leading us to believe that the large-scale spatial structure is Euclidean after all, we cannot now go back to flatness for the structure on a more localized level. Tests of general relativity are now sufficiently good to provide direct measurements of deviations from flatness in physical geometry, the geometry being defined in a sense that I shall describe in the next section. However, it is the geometry of *space-time*, rather than that of space, which has a clear physical interpretation. And we shall see that the space-time geometry of (Poincaré's!) *special* relativity, though flat, is yet *not* the geometry of Euclid.

In §2, I shall outline the contemporary and now well-established (Einstein) view of curved space-time geometry. Then in §3, I shall indicate how this geometry may be looked at in a different way, and how this new viewpoint enables another of Poincaré's pioneering innovations, namely the study of the instrinsic structure of boundaries of complex domains, to be applied in an unexpected context in physics. Finally, in §4, I shall indicate how, in a sense, these ideas also enable the physics to repay its debt to the mathematics and provide what appears to be the first established example of a complex manifold which cannot be locally extended beyond its C^{∞}-smooth boundary.

2. Structure of space-time. It is to Minkowski (1908) whom we owe the idea that physical geometry should be a 4-dimensional space-time geometry, rather than a 3-dimensional spatial one. In relativity theory, there is no absolute 3-geometry, but the 4-geometry of space-time is an absolute objective physical structure. In fact, when viewed in the Minkowskian way, relativity theory becomes a theory of the absolute. What have become relative are merely the older ideas of separate space and time. One further point about Minkowski's absolute space-time geometry that will emerge is that it is most directly described in terms of measured time-intervals rather than of distances in the ordinary sense, so that the geometry is really a "chronometry". This point of view has been emphasized particularly by Synge (1960) and Bondi (1965) and it leads to a considerable clarification of the meanings of the mathematical structures involved.

In order to appreciate fully the nature of the space-time provided by Einstein's general relativity, it may be helpful first to see how the older physical theories can be described in a space-time setting. We shall consider, in turn, five alternative space-time structures, which I shall refer to (cf. Penrose (1968)) as

A: Aristotelean space-time,
G: Galilean space-time,
N: Newtonian space-time,
M: Minkowskian space-time,
E: Einsteinian space-time.

The structure of the Aristotelean space-time A is given by its expression as a product

$$A = \mathbb{E}^1 \times \mathbb{E}^3, \tag{2.1}$$

where $\mathbb{E}^n$ denotes Euclidean n-space equipped with its flat metric (and, if desired, with an orientation), whence each $\mathbb{E}^n$ possesses a $\frac{1}{2}n(n+1)$ group of symmetries. For any two events $a, b \in A$, there is both an absolute time-difference $T(a, b) \in \mathbb{R}$ and an absolute spatial separation $D(a, b) \in \mathbb{R}$ (these being the distance functions in $\mathbb{E}^1$ and $\mathbb{E}^3$, respectively). It is therefore meaningful to say, of two events, whether they are simultaneous in A (i.e. $T(a, b) = 0$); but it is also meaningful to say whether they have the same position (i.e. $D(a, b) = 0$) irrespective of their simultaneity. For in Aristotelean physics the state of rest is distinguished among all motions, and this is reflected in the fact that the canonical copies of $\mathbb{E}^1$ in

$\mathbb{E}^1 \times \mathbb{E}^3$ constitute a distinguished family of curves in A representing the world-lines of particles at rest, i.e. which have the same position at all times. The space A has a 7-parameter symmetry group.

The structure of Galilean space-time G, on the other hand is not that of a product but merely a fibre bundle with $\mathbb{E}^3$ fibres and an $\mathbb{E}^1$ base space, the projection map

$$G \to \mathbb{E}^1 \tag{2.2}$$

being the assignment of a "time" to any event in G. Thus, of any two events $a, b \in G$, it is meaningful to speak of their time-difference $T(a, b) \in \mathbb{R}$ and to say when the events are simultaneous (i.e. $T(a, b) = 0$), but their spatial separation *only* has meaning if they *are* simultaneous. This reflects the fact that in Galilean physics there is no invariantly defined state of rest, and any two nonsimultaneous events will be viewed as having the same spatial location with respect to *some* suitably moving reference system. The bundle structure of G is not, however, G's whole structure. We need also to single out a family of curves in G which represent *inertially moving particles*; and these curves may be referred to as *straight lines* in G. The simplest way to specify the particular straight line structure that G possesses is, perhaps, to say that G is an *affine space* and that the fibration (2.2) is *compatible* with this affine structure (in the sense that the fibres are all parallel, that the $\mathbb{E}^3$ fibres and $\mathbb{E}^1$ base all inherit their correct affine structures, and that parallel straight lines, transverse to the $\mathbb{E}^3$'s, provide metric-preserving maps between the $\mathbb{E}^3$'s). The total structure that all this assigns to G is somewhat weaker than the structure of A. The symmetry group of G contains that of A as a subgroup, and it is a 10-parameter group referred to as the *Galilei group*.

We pass now to Newtonian space-time N, the idea here (due to É. Cartan (1923), (1924), cf. also Friedrichs (1928), Trautman (1966)) being to treat Newtonian gravitational theory as a geometric theory in the spirit of Einstein's general relativity. We recall, for example, that a *uniform* Newtonian gravitational field permeating the whole of space would be totally undetectable. All particles would accelerate together under this field, so that relative to the particles themselves, the field would appear not to be there at all. It is only deviations from uniformity, i.e. "tidal forces", which are physically detectable, and the idea is that in the general case these supply a kind of curvature for N. As a fibre bundle, N has a structure identical with that for G,

$$N \to \mathbb{E}^1 \tag{2.3}$$

(so simultaneity and time-difference, and lack of a spatial separation concept for nonsimultaneous events, are the same as before)—but now there is a different family of curves representing inertial motions. "Inertial motion" now means "free motion under gravity", and we can refer to the corresponding world-lines in N as "geodesics". In fact, while not extremal curves, these geodesics are, indeed, the geodesics of a certain torsion-free linear connection Γ on N. The space N also possesses a *co*metric $\mathbf{g}^*$ (symmetric contravariant 2-valent tensor) which is semi-definite but degenerate (having matrix-rank 3), where $\mathbf{g}^*$ has the defining property of inducing the correct Euclidean metric on each $\mathbb{E}^3$ fibre. The connection Γ preserves $\mathbf{g}^*$, but this property does not define Γ uniquely. The different possible inequivalent Γ's correspond to the different possible physically inequivalent gravitational fields. Thus, unlike G and A, which have canonical structures, there are many *different* Newtonian space-times N. Indeed, G is a special case, given when Γ is flat. In general, N has no symmetries.

Minkowskian space-time M, the space-time of special relativity, on the other hand, possesses, like G and A, a canonical and uniform structure. Like G (and like A), M is an affine space, and inertial motions in M are described by lines which are straight with respect to this affine structure. These lines may also be thought of as the geodesics of a certain flat connection Γ, where Γ is now the unique torsion-free connection preserving a certain flat nondegenerate pseudometric $\mathbf{g}$, of Lorentzian signature $(+,-,-,-)$. (It is the nondegeneracy of $\mathbf{g}$ that now enables it to determine Γ uniquely.) The pseudometric $\mathbf{g}$ serves to define a quadratic squared "distance" function S of signature $(+,-,-,-)$, on pairs $a, b \in M$ ($S(a, b) \in \mathbb{R}$). Thus M is a pseudo-Euclidean space. The physical meaning of S is that an (ideal) inertial (i.e. unaccelerated) clock which moves from the event a to the event b will register a time-interval between a and b equal to $\{S(a, b)\}^{1/2}$. For a physical clock (or, indeed, for any massive particle) $S(a, b) > 0$, and we say that the separation between a and b is *timelike*. When $S(a, b) = 0$ we say that the separation is *null*, this being the case when a light ray connects a to b. When $S(a, b) < 0$ we say that the separation is *spacelike*, and in this case a and b will appear as simultaneous in some suitably moving inertial reference system, the distance between them (using units for which the light-speed is unity) being $\{-S(a, b)\}^{1/2}$. The symmetry group of M is the 10-parameter *Poincaré group*.

The structure of Einsteinian space-time E bears the same relation to M as N does to G. Thus there are many inequivalent spaces E, their different structures providing the various inequivalent gravitational fields. The structure, in each case, is given by a pseudometric $\mathbf{g}$, with the same $(+,-,-,-)$ signature as M, but now $\mathbf{g}$ is generally not flat. The inertial motions in E (i.e. free motions under gravity) are given, as with M, by geodesics of the unique torsion-free connection Γ preserving $\mathbf{g}$. But as with N, a *curvature* of Γ provides the physically detectable "tidal force". As with M, the pseudometric $\mathbf{g}$ provides the definition of time-interval as measured by ideal clocks, but because $\mathbf{g}$ is not flat, we normally think of the interval as defined only between infinitesimally separated points, the interval (in the timelike or null case) being given by $\{\mathbf{g}(\delta x, \delta x)\}^{1/2}$, where δx is the tangent vector which "connects" the point to its infinitestimally separated neighbour. The world-line of a massive particle has tangent vectors which are all timelike (i.e. $\mathbf{g}(\delta x, \delta x) > 0$), the tangent vectors being all null (i.e. $\mathbf{g}(\delta x, \delta x) = 0$) in the case of a massless particle. The time-interval between finitely-separated events a and b, as measured by a clock carried from a to b by such a particle is the pseudo-Riemannian "distance", given by

$$\int_a^b \{\mathbf{g}(dx, dx)\}^{1/2}.$$

Note that this time-interval depends upon the path through space-time from a to b. If a and b are not too far apart, this time-interval is a *maximum* for the geodesic (i.e. inertial path) which locally connects a to b. The metric $\mathbf{g}$ determines (via Γ) a Riemann curvature tensor $\mathbf{R}$. The direct physical interpretation of $\mathbf{R}$ is obtained from the Jacobi equation, which describes, in terms of $\mathbf{R}$, the geodesic derivation of (say) timelike geodesics, i.e. the tidal effect on inertially moving particles. In the general case, E has no symmetries.

In order to provide a physical theory of gravity, Einstein had to supplement the above general geometric structure by his *field equations*. These state that the trace-reversed Ricci tensor constructed from $\mathbf{R}$ is a constant multiple of the physical energy-momentum tensor, describing the density of matter. Where there is no matter, the energy-momentum tensor is zero and the space-time is Ricci-flat; and where matter is present, the Ricci tensor is locally determined by this matter density. The remainder of the curvature, namely that measured by the Weyl conformal curvature tensor $\mathbf{C}$, is governed by differential equations and is thus constrained, in a nonlocal way, by the distribution of matter. In the limit when velocities are small and the fields are weak (in the sense of small gravitational potentials) the theory goes over into the older Newtonian theory. Various small deviations from Newtonian theory have been observationally established, and these are all consistent with the more accurate Einstein theory. In fact, Einstein's theory must now be regarded as an excellently-tested theory of gravity, having no serious rival (at least none for which there is any forseeable prospect of observing a discrepancy with Einstein's theory). The mathematics of Einsteinian manifolds —and including that of the important special case provided by Minkowski geometry—is thus something of especial interest for physics.

3. Twistor geometry and CR-structure. Let us next examine how the geometry of Minkowski space M may be reformulated in a new way (cf. Penrose (1975), (1977); Penrose and Ward (1980); also Wells (1979)). We shall see how this relates M to the theory of boundaries of complex manifolds. An analogous construction for the more general curved Einstein spaces will be considered afterwards.

Choose standard Minkowski coordinates $x = (\tau, \xi, \eta, \zeta) \in \mathbb{R}^4$, for M, where S is given by

$$\begin{aligned} S(x, \hat{x}) &= (\tau - \hat{\tau})^2 - (\xi - \hat{\xi})^2 - (\eta - \hat{\eta})^2 - (\zeta - \hat{\zeta})^2 \\ &= \det\left\{\begin{pmatrix} \tau + \zeta & \xi + i\eta \\ \xi - i\eta & \tau - \zeta \end{pmatrix} - \begin{pmatrix} \hat{\tau} + \hat{\zeta} & \hat{\xi} + i\hat{\eta} \\ \hat{\xi} - i\hat{\eta} & \hat{\tau} - \hat{\zeta} \end{pmatrix}\right\} \end{aligned}$$

where throughout this work "i" stands for $\sqrt{-1}$ (and a bar denotes complex conjugation).

Let $(W_0, W_1, W_2, W_3) \in \mathbb{C}^4$ be coordinates for an element W of an associated complex vector space $\mathbb{T}.$ called (dual) twistor space where W and x are said to be *incident* whenever

$$(W_2, W_3) = \frac{1}{i\sqrt{2}}(W_0, W_1)\begin{pmatrix} \tau + \zeta & \xi + i\eta \\ \xi - i\eta & \tau - \zeta \end{pmatrix}. \tag{3.2}$$

This relation can only hold if $\Sigma(W) = 0$, where the $(+,+,-,-)$-signature Hermitian form Σ is defined by

$$\Sigma(W) \equiv W_0\overline{W_2} + W_1\overline{W_3} + W_2\overline{W_0} + W_3\overline{W_1} \tag{3.3}$$

(because the Hermiticity of the (2×2) matrix in (3.2) entails that postmultiplication of (3.2) by the conjugate transpose of (W_0, W_1) yields a purely imaginary result).

The 3-complex-dimensional projective space $\mathbb{PT}.$, associated with $\mathbb{T}.$, whose points are labelled by the three complex ratios

$$W_0 : W_1 : W_2 : W_3$$

consists of two open complex manifolds $\mathbb{PT}_+$ and $\mathbb{PT}_-$, given when $\Sigma(W) > 0$ and $\Sigma(W) < 0$, respectively, together with their common boundary, the 5-real-dimensional manifold $\mathbb{PT}_0$, given when $\Sigma(W) = 0$. Thus the points of $\mathbb{PT}.$ for which (3.2) holds for some $x \in M$, all lie on $\mathbb{PT}_0$. They do not constitute quite the whole of $\mathbb{PT}_0$, however, since points of the projective line I, given by $W_0 = W_1 = 0$, admit no solution for x in (3.2).

Suppose we choose a fixed

$$\mathbf{W} \in \mathbb{PT}_0 - I$$

(where from now on I use boldface capital letters such as $\mathbf{W}$ to denote the point of the projective space $\mathbb{PT}.$, rather than $\mathbb{T}.$ or $\mathbb{C}^4$). Then we can solve (3.2) for the Minkowski point x. The solution is not unique, however, the freedom being given by

$$\begin{pmatrix} \tau + \zeta & \xi + i\eta \\ \xi - i\eta & \tau - \zeta \end{pmatrix} \mapsto \begin{pmatrix} \tau + \zeta & \xi + i\eta \\ \xi - i\eta & \tau - \zeta \end{pmatrix} + k\begin{pmatrix} W_1\overline{W_1} & -W_1\overline{W_0} \\ -W_0\overline{W_1} & W_0\overline{W_0} \end{pmatrix}$$

for arbitrary real k. Note that the final matrix on the right has rank unity and so represents, by (3.1), a *null* (i.e. zero Minkowski length) vector in M. Thus, the points x incident with the given W constitute a *null geodesic* (straight line) in M which, without confusion, we can also label by $\mathbf{W}$. Indeed, we may think of $\mathbb{PT}_0 - I$ *as* the space of null geodesics $\mathbf{W}$ in M. Our construction has shown how this space may be imbedded as a 5-dimensional real submanifold of a complex projective 3-space.

To interpret a point $x \in M$, conversely in terms of $\mathbb{PT}.$, we fix x in (3.2) and allow $\mathbf{W}$ to vary. This gives us a 2-complex-dimensional linear subspace in $\mathbb{T}.$, i.e.

a complex projective line in $\mathbb{PT}.$. Clearly this line lies in $\mathbb{PT}_0 - I$ and, moreover, all lines lying entirely in $\mathbb{PT}_0 - I$ arise in this way from points in M. Thus

the projective lines lying entirely in $\mathbb{PT}_0 - I$
represent the points of Minkowski space M.

Note that if we allow the coordinates τ, ξ, η, ζ to become complex i.e. x to become a point of the *complexification* $\mathbb{C}M$ of M, then the above construction yields a projective line lying entirely in $\mathbb{PT}. - I$. We may also consider projective lines in $\mathbb{PT}_0$ [resp. $\mathbb{PT}.$] which meet I. These provide "points at infinity" for M [resp. $\mathbb{C}M$], the totality of all lines in $\mathbb{PT}_0$ [resp. $\mathbb{PT}.$] describing the standard conformal compactification $M^{\#}$ [resp. $\mathbb{C}M^{\#}$] of M [resp. $\mathbb{C}M$].

The intrinsic structure of projective lines in $\mathbb{PT}_0 - I$ can be illustrated in a very graphic "physical" way. The family of light rays **W** through (i.e. incident with) a fixed point $x \in M$ represents the family of points on the corresponding complex projective line in $\mathbb{PT}_0 - I$. A complex projective line has the topology S^2 and, moreover, the holomorphic structure of a Riemann sphere. Now imagine an observer situated at x. His field of vision will be represented by the light rays through x, i.e. by the family of **W**'s under consideration. Clearly the topology of the observer's entire field of vision is indeed S^2. But, more subtly, the *holomorphic* structure is also relevant. This shows up in the allowed transformations between various observers with different velocities, all of whom pass through the same space-time event x. Their fields of vision are related to one another, at x, by transformations which preserve the holomorphic structure of the Riemann sphere, i.e. their fields of vision are *conformally* related to one another. This is a well-known property of Lorentz transformations (Penrose (1959), Terrell (1959)). Indeed, the connected component of the *Lorentz group* may be regarded as the group of all *holomorphic self-transformations* of this Riemann sphere.

By virtue of its imbedding in the complex manifold $\mathbb{PT}.$, the entire (real) hypersurface $\mathbb{PT}_0$ inherits further holomorphic structure. This structure, referred to as a (maximal[1]) CR-structure falls short by just one (real) dimension of defining $\mathbb{PT}_0$ as a complex manifold. We shall see the physical interpretation of this shortly.

The structure of a (maximal) CR-manifold $\mathcal{V}$ of, say, dimension $2n + 1$ provides that in the tangent space T_p, at each point $p \in \mathcal{V}$, a $2n$-real-dimensional subspace $H_p \subset T_p$ is singled out, referred to as the *holomorphic tangent space* at p. The space H_p is to be regarded as a *complex* vector space of n dimensions, spanned by complex vectors

$$\mathbf{z}_1 = \mathbf{x}_1 + i\mathbf{y}_1, \ldots, \mathbf{z}_n = \mathbf{x}_n + i\mathbf{y}_n.$$

[1]Some authors use the term "CR-structure" also in the extended sense of the structure induced on any real submanifold of a complex manifold, not necessarily a hypersurface. The word "maximal" is to emphasize that I am concerned only with the hypersurface dimensionality here.

The real-linear operator J, satisfying $J^2 = -1$, is determined by its action on the related real basis $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_n$ for H_p by

$$(3.4) \qquad J\mathbf{x}_r = -\mathbf{y}_r, \qquad J\mathbf{y}_r = \mathbf{x}_r \qquad (r = 1, \ldots, n)$$

so that

$$(3.5) \qquad J\mathbf{z}_r = i\mathbf{z}_r \qquad (r = 1, \ldots, n).$$

It is the operator J that defines the complex structure of H_p when we regard H_p as a real vector space. One further tangent vector $\mathbf{u} \in T_p$, independent of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, $\mathbf{y}_1, \ldots, \mathbf{y}_n$, is needed to complete a basis for the whole of T_p. We assume that in some neighbourhood $\mathcal{N}_p$ of p in $\mathcal{V}$ a smooth choice of such basis vectors is made and we define the *integrability relations* for the CR-structure of $\mathcal{V}$, defined by J, to be the Lie bracket relation

$$(3.6) \qquad [\mathbf{z}_r, \mathbf{z}_{r'}] = \text{complex linear combinations of } \mathbf{z}\text{'s}$$

$(r, r' = 1, \ldots, n)$. It is clear that this integrability relation is independent of the particular choice of basis satisfying (3.5), and depends only on the choice of J and the holomorphic subspaces.

Note that what distinguishes the structure of $\mathcal{V}$ from that of a complex manifold (of dimension n) is merely the presence of the extra real dimension defined by the extra real basis vector $\mathbf{u}$.

In the case of a *complex manifold* we know from the Newlander-Nirenberg (1957) theorem that integrability relations like (3.6) are all that are required. No analyticity assumptions are needed to ensure that holomorphic coordinates can be locally introduced, the derivatives with respect to which providing complex tangent vectors compatible with the given complex structure J in the sense of (3.5). The analogous property for a (maximal) CR-manifold $\mathcal{V}$ of real dimension $2n + 1$ would be the existence of a complex $(n + 1)$-manifold $\mathcal{C}$, with complex structure J, in which $\mathcal{V}$ can be realized as a real hypersurface, the CR-structure on $\mathcal{V}$ being that induced by J. However, counterexamples given by Nirenberg [(1973), (1974)] show that the integrability relations (3.6) are insufficient to ensure that the CR-structure of $\mathcal{V}$ can be so realized in all cases.

It should be made clear that the difficulty does not lie in a possible omission of necessary additional integrability relations of the usual (differential) kind. Indeed, if the CR-structure is taken to be real-analytic, then the integrability relations (3.6) are actually sufficient to ensure that $\mathcal{V}$ can be realized in the above way. But for a C^∞ CR-structure this need not be so. In this respect, the problem of realizing CR-structures is analogous to that of the Lewy insoluble differential equation (1957). We shall see later how this can be made more explicit.

Let us return to $\mathbb{PT}_0$, which has a CR-structure explicitly realized by its imbedding in the complex manifold $\mathbb{PT}$.. Here $n = 2$. Thus, at any point $\mathbf{W}$ of $\mathbb{PT}_0$ the 5-real-dimensional tangent space $T_\mathbf{W}$ has a 2-complex-dimensional holomorphic subspace $H_\mathbf{W}$ whose complete structure is defined by a certain operator J. Let us first identify the location of $H_\mathbf{W}$ in physical terms. The point $\mathbf{W} \in \mathbb{PT}_0$ is interpreted as a light ray in M. We may picture this by fixing the "time" τ to take a specific value, say $\tau = 0$, so our picture becomes that of ordinary Euclidean 3-space U. Then the "photon" described by $\mathbf{W}$ is pictured as a *point* in U (its location at $\tau = 0$) together with a *direction* at that point defining its velocity (the magnitude of the velocity being unity: the "speed of light"), i.e. we represent $\mathbf{W}$ by a unit vector $\mathbf{w}$ at some point q of U. To picture $T_{\mathbf{W}}$ we envisage making a small displacement of q and $\mathbf{w}$. If this displacement is such that q is moved in a direction orthogonal to the direction of $\mathbf{w}$, then we get something in the subspace $H_{\mathbf{W}}$. We refer to these slightly displaced light rays as being *abreast* with one another. The subspace $H_{\mathbf{W}}$ divides $T_{\mathbf{W}}$ into two remaining pieces, representing light rays that are slightly ahead or lag slightly behind the original ray $\mathbf{W}$. These properties are easily seen to be independent of the "time" τ, i.e. independent of the particular Euclidean (spacelike) hyperplane U that is chosen for our description.

To picture the effect of J, imagine a 2-plane element π, at q, which is orthogonal to the direction $\mathbf{w}$. We are concerned only with displacements of q and $\mathbf{w}$ for which q is moved within π, since these correspond to real vectors in $H_{\mathbf{W}}$ (i.e. "abreast" displacements of $\mathbf{W}$). Such displacements are represented by pairs of vectors $(\mathbf{r}, \mathbf{v})$ in π, where $\mathbf{r}$ gives the displacement of q and $\mathbf{v}$ measures the change in $\mathbf{w}$ (clearly also orthogonal to $\mathbf{w}$, since $\mathbf{w}$ is a unit vector). If we keep the light ray $\mathbf{W}$ and the "neighbouring" ray to which it is displaced both fixed, but vary the time τ, we easily find that the functions $\mathbf{r}(\tau)$, $\mathbf{v}(\tau)$ are related by

(3.7) $$d\mathbf{r}/d\tau = \mathbf{v}, \qquad d\mathbf{v}/d\tau = 0.$$

Now the effect of J turns out to be simply to *rotate both* $\mathbf{r}$ *and* $\mathbf{v}$ *through a right angle in the plane* π, in a left-handed sense about the direction of $\mathbf{w}$. Clearly (3.7) is invariant under this operation, so J applies to the displacements of light rays in their entirety and not simply to light rays relative to particular points on them. (Moreover this operation is easily seen to be independent of the slope of the hyperplane U.) Thus, the CR-structure of $\mathbb{P}\mathbb{T}_0$ is seen to have direct interpretation in terms of physical space-time geometry.

Let us next consider what happens when Minkowski space M is replaced by a more general Einsteinian space-time manifold $\mathfrak{M}$. We shall suppose, for simplicity that $\mathfrak{M}$, with its pseudometric $\mathbf{g}$, is *globally hyperbolic* (Leray (1952)). According to a result of Geroch (1970) this property can be stated as the existence of a spacelike hypersurface $\mathfrak{U}$ in $\mathfrak{M}$ which intersects every null geodesic (light ray) in $\mathfrak{M}$ in precisely one point. ("Spacelike", in this context, means that its normal vectors are everywhere timelike, i.e. that its induced metric is everywhere negative definite.)

It turns out that the space $\mathbb{P}\mathfrak{T}_0$ of null geodesics in $\mathfrak{M}$ acquires a CR-structure relative to the hypersurface $\mathfrak{U}$, where the holomorphic subspaces to the tangent spaces in $\mathbb{P}\mathfrak{T}_0$ and the operator J, now denoted $J_{\mathfrak{U}}$, are defined essentially as before. Thus we may represent any $\mathbf{W} \in \mathbb{P}\mathfrak{T}_0$ by the point $q \in \mathfrak{U}$ at which $\mathfrak{U}$ intersects the null geodesic $\mathbf{W}$, together with the unit vector $\mathbf{w}$ in $\mathfrak{U}$ at q in the direction of the orthogonal projection into $\mathfrak{U}$ of the future-pointing null direction of **W**. The holomorphic tangent space to $\mathbb{P}\mathfrak{T}_0$ at **W** is provided by those displacements of **W** for which q moves in a direction orthogonal to **w**, and the (real) vectors in this space therefore correspond to pairs (**r**, **v**), as before, where **r** and **v** are (real) tangent vectors to $\mathcal{U}$ at q lying in the 2-plane element π orthogonal to **w**. The effect of $J_{\mathcal{U}}$ is, as before, to rotate **r** and **v** through a right angle, in a left-handed sense about **w**. It then turns out that the integrability relations (3.6) for $J_{\mathcal{U}}$ are automatically satisfied (Penrose (1975), LeBrun (1980)) and the required CR-structure, denoted $\mathrm{CR}_{\mathcal{U}}$, is thereby obtained.

However, unlike the CR-structure for $\mathbb{PT}_0$, $\mathrm{CR}_{\mathcal{U}}$ will in general depend upon the location of $\mathcal{U}$ in $\mathcal{M}$. The holomorphic tangent spaces, regarded as real vector subspaces of the tangent spaces to $\mathbb{P}\mathfrak{T}_0$, are in fact independent of $\mathcal{U}$, but their complex structures, as defined by $J_{\mathcal{U}}$ will generally vary. These properties follow from the Jacobi equation, which suitably replaces the second of equations (3.7). Only when the metric **g** of $\mathcal{M}$ is conformally flat will $\mathrm{CR}_{\mathcal{U}}$ be completely independent of $\mathcal{U}$.

By way of clarification of this point it may be remarked that even if $\mathcal{M}$ is *stationary* (i.e. time-independent, in the sense of possessing a timelike killing vector), so that isometries of $\mathcal{M}$ exist carrying $\mathcal{U}$ into a succession of geometrically equivalent spacelike hypersurfaces $\mathcal{U}'$, $\mathcal{U}''$, $\ldots$, $\subset \mathcal{M}$, the structures $\mathrm{CR}_{\mathcal{U}}$, $\mathrm{CR}_{\mathcal{U}'}$, $\mathrm{CR}_{\mathcal{U}''}$, $\ldots$ on $\mathbb{P}\mathfrak{T}_0$ will generally be distinct. This is because a particular point **W** of $\mathbb{P}\mathfrak{T}_0$ represents a nonstationary object (namely a null geodesic) in $\mathcal{M}$, so it is related differently to each of $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{U}''$, $\ldots$. However, in this case the isometries of $\mathcal{M}$ will carry **W** into null geodesics **W**$'$, **W**$''$, $\ldots$, so $\mathrm{CR}_{\mathcal{U}}$ at **W** will agree with $\mathrm{CR}_{\mathcal{U}'}$, at **W**$'$ and with $\mathrm{CR}_{\mathcal{U}''}$ at **W**$''$, etc. Thus the structures $\mathrm{CR}_{\mathcal{U}}$, $\mathrm{CR}_{\mathcal{U}'}$, $\ldots$ will in this case be *intrinsically* identical even though they represent generally distinct CR-structures on the given space $\mathbb{P}\mathfrak{T}_0$.

The fact that CR-structures (maximal, and of given dimension $2n + 1 > 0$) *can* be locally *distinct* from one another is, in effect, an observation that dates back to an important paper of Poincaré (1907). What Poincaré actually showed was that the Riemann mapping theorem, which states that any smoothly bounded simply-connected region $\mathcal{D}$ in the Argand plane $\mathbb{C}^1$ is holomorphically identical with the unit disc, has no direct analogue in higher complex dimension. Roughly, the argument is to show that the (smooth) boundary of a region in, say, $\mathbb{C}^2$ contains intrinsic holomorphically invariant information about its "shape". (This does not occur for a smooth curve in $\mathbb{C}^1$, as the Riemann mapping theorem shows.) The gist of Poincaré's argument can be simplified to the following "physicists proof". Consider the freedom involved in specifying the (smooth) real hypersurface boundary of a region in $\mathbb{C}^2$. This is provided by one real function of three real variables (e.g. $\mathrm{Im}\,\zeta_1$ in terms of $\mathrm{Re}\,\zeta_1$, $\mathrm{Re}\,\zeta_2$, $\mathrm{Im}\,\zeta_2$). For the *intrinsic* structure of this boundary, we must factor out by the freedom provided by the allowed local holomorphic maps of $\mathbb{C}^2$ to itself. This is provided (locally) by two holomorphic functions of two complex variables. But a holomorphic function is determined by its (analytic) values on any real environment, e.g. on its values where ζ_1 and ζ_2 are real. The real and imaginary parts there are each, in effect, independent real analytic functions, so the freedom to be factored out by is that of four real functions of two real variables. The amount of intrinsic freedom in the structure of the boundary is therefore

$$\frac{\text{1 real function of 3 real variables}}{\text{4 real functions of 2 real variables}}.$$

However, any finite number of functions of two variables must be regarded as "peanuts" in the context of free functions of three variables, i.e. it is completely swamped by the three-variables' worth of freedom and makes no contribution net count. Thus there must be "intrinsic invariants" of the boundary shape whose functional freedom is such as to be dependent upon three real variables. More generally, for a boundary in $\mathbb{C}^{n+1}$ (and so for a $(2n + 1)$-dimensional CR-manifold) the freedom for the invariants is functions of $2n + 1$ variables ($n > 0$).

The detailed form of these invariants was investigated thoroughly by É. Cartan (1932), Tanaka (1962) and Chern & Moser (1974). This study may be regarded as the analogue, for CR-manifolds, of the study of invariants (and covariant tensorial objects) for ordinary Riemannian geometry.

It turns out that the structures $\mathrm{CR}_{\mathcal{U}}$ that we have just been considering do in fact differ, in general, from the original CR-structure of $\mathbb{PT}_0$ that arose in relation to Minkowski space. There is the further point that by choosing $\mathcal{M}$ and $\mathcal{U}$ to be suitably nonanalytic, the structure $\mathrm{CR}_{\mathcal{U}}$ can be made to be *nonrealizable* as a real hypersurface in a complex 3-manifold, analogously to the Nirenberg counterexamples referred to earlier. This was first demonstrated by C. R. LeBrun (1980), (1982). Some of the relevant ideas and a slightly different, but related, class of counterexamples had been put forward earlier by G. A. J. Sparling (and involved a suggestion by the present author). A development of Sparling's original line of thinking will be presented in the next section.

4. Boundaries of cohomology classes and complex manifolds. The first and most important invariant of CR-structure is the signature of the Levi form. One way of defining the Levi form for a $(2n + 1)$-dimensional CR-manifold $\mathcal{V}$ is as follows. Suppose $\mathcal{V}$ is given, locally, as a real hypersurface

$$\Sigma(\zeta_0,\ldots,\zeta_n) = 0$$

in a complex n-manifold $\mathcal{C}$, where Σ is a real (at least C^2) function of local holomorphic coordinates $\zeta_0,\ldots,\zeta_n$ for $\mathcal{C}$. Then the Hermitian form whose matrix is

$$\frac{\partial\Sigma}{\partial\zeta_j\partial\bar{\zeta}_k},$$

but restricted to the holomorphic tangent space H_p, defines the Levi form at p. Alternatively an entirely intrinsic definition can be given which does not depend upon an imbedding in a complex $(n + 1)$-manifold and so applies to nonrealizable CR-structures. Let the complex basis vectors $\mathbf{z}_1,\ldots,\mathbf{z}_n$ and real basis vector $\mathbf{u}$

be as in §3 (cf. (3.5), (3.6)). Then the Lie brackets

$$[\mathbf{z}_j, \bar{\mathbf{z}}_k] = iL_{jk}\mathbf{u} + \text{terms in } \mathbf{z}\text{'s and } \bar{\mathbf{z}}\text{'s} \tag{4.1}$$

serve to define the matrix L_{jk} of the Levi form intrinsically.

Though the matrix of the Levi form is clearly not an invariant, its signature—by which I mean the number of plus signs, minus signs and zeros that it acquires when unitarily reduced to diagonal form—is invariant. A remark about the *sign* of the Levi form should also be made here. Suppose we are concerned with the (smooth) boundary $\mathcal{V}$ of a complex manifold; then we should take that complex manifold on the side on which Σ is negative. Thus, for example, for the unit ball in $\mathbb{C}^{n+1}$, we can take

$$\Sigma = \zeta_0\bar{\zeta}_0 + \cdots + \zeta_n\bar{\zeta}_n - 1$$

and the Levi form is positive definite $(+ + \cdots +)$, this being the situation in which the manifold is referred to as (holomorphically) strictly *pseudoconvex*. Correspondingly, if we use the intrinsic method (4.1), we choose the vector $\mathbf{u}$ so that $J\mathbf{u}$ points *outwards* away from the manifold whose boundary we are considering. Generally, if the Levi form signature has j plus signs and k minus signs at a point of $\mathcal{V}$, I shall say that this manifold has j degrees of pseudoconvexity and k degrees of pseudoconcavity at p.

An important result, effectively due to Hans Lewy (1956) (and Bochner (1943); cf. also Hörmander (1966)) states if f is a CR-function on $\mathcal{V}$—which means that

$$\bar{z}_1(f) = 0, \ldots, \bar{z}_n(f) = 0$$

—where we assume that $\mathcal{V}$ is realizable as a real hypersurface in a complex manifold $\mathcal{C}$, then f extends (locally) as a holomorphic function on any side of $\mathcal{V}$ having at least one degree of pseudoconvexity.

It should be remarked that any holomorphic function in $\mathcal{C}$ clearly restricts to a CR-function on $\mathcal{V}$. Moreover, any holomorphic function on one side of $\mathcal{V}$ which attains smooth values at $\mathcal{V}$ itself will also restrict to a CR-function on $\mathcal{V}$. The (Lewy) result just stated shows that a holomorphic function defined on a smoothly bounded region $\mathcal{D}$ of a complex manifold is always locally extendible beyond that boundary at points where $\mathcal{D}$ has at least one degree of pseudoconcavity. Thus smoothly bounded domains of holomorphy, on which locally inextendible holomorphic functions exist, have no points with any degree of pseudoconcavity—i.e. the Levi signature is always of the form $(+ \cdots +0 \cdots 0)$.

Let us now consider the 5-dimensional real hypersurface $\mathbb{PT}_0$ as a common boundary between $\mathbb{PT}_-$ and $\mathbb{PT}_+$. By simple direct calculation from (3.3), we find that the Levi form signature is $(+,-)$, while for the 7-dimensional real hypersurface $\mathbb{T}_0$ as a boundary between $\mathbb{T}_-$ and $\mathbb{T}_+$ the signature is $(+,-,0)$. (The zero arises from the holomorphic direction corresponding to complex rescalings $W \mapsto \lambda W$, $\lambda \in \mathbb{C} - \{0\}$.) From the results just stated we see that any CR-function defined in some open neighbourhood of a point in $\mathbb{PT}_0$ [resp. $\mathbb{T}_0$] will extend to a holomorphic function in a neighbourhood in $\mathbb{PT}$. [resp. $\mathbb{T}$.].

It turns out, however, that the most directly physical objects on these spaces are not holomorphic or CR-functions but objects of the next degree of abstraction, namely *first sheaf cohomology classes*. The simplest example of the physical interpretation of such a cohomology class can be illustrated as follows. Consider some suitable open neighbourhood $\mathcal{Q}$, in $\mathbb{C}M$, of a point $x \in \mathbb{C}M$. According to the discussion of §3, each point of $\mathcal{Q}$ will be represented by a projective line in $\mathbb{PT}_\bullet$, so as the point varies throughout $\mathcal{Q}$ the corresponding line sweeps out an open region $\mathcal{L} \subset \mathbb{PT}_\bullet$. We shall be concerned with an element

$$\Phi \in H^1(\mathcal{L}, \mathcal{O}(-2))$$

where, generally, $\mathcal{O}(r)$ denotes the sheaf of germs of holomorphic functions on $\mathbb{PT}_\bullet$ "twisted by r"—i.e. represented in terms of holomorphic functions on $\mathbb{T}_\bullet$ which are *homogeneous of degree r*. Under fairly liberal restrictions on the nature of $\mathcal{Q}$ (cf. Eastwood, Penrose and Wells (1981) for details) we find that Φ represents a solution ϕ of the wave equation

(4.2) $$\Box\phi \equiv \left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\zeta^2}\right)\phi = 0$$

in $\mathcal{Q}$.

Suppose we have the simple situation in which Φ can be described by a representative Čech cocycle Φ_{12} $(= -\Phi_{21})$ for the 2-set cover $\mathcal{N}_1, \mathcal{N}_2$ of $\mathcal{L}$, so Φ_{12} is simply a holomorphic function (taken as homogeneous of degree -2 in $W_0, \ldots, W_3$) on $\mathcal{N}_{12} = \mathcal{N}_1 \cap \mathcal{N}_2$, where $\mathcal{L} \subset \mathcal{N}_1 \cup \mathcal{N}_2$. To obtain ϕ, we then perform the contour integral

(4.3) $$\phi(x) = \frac{1}{2\pi i}\oint_{\Xi} \Phi_{12}(W)\Delta W$$

with

(4.4) $$\Delta W = W_0 dW_1 - W_1 dW_0$$

where the 1-dimensional closed contour Ξ (in $\mathbb{PT}_\bullet$) lies in the intersection of $\mathcal{N}_{12}$ with the projective line $L_x \subset \mathbb{PT}_\bullet$ representing $x = (\tau, \xi, \eta, \zeta)$. Note that since we are integrating over points of $\mathbb{PT}_\bullet$ *incident* with x, we can use the explicit expression (3.2) to represent W_2 and W_3 in terms of W_0 and W_1. Moreover, because of the -2 homogeneity of Φ_{12} and the form of the expression (4.4), the exterior derivative of the integrand in (4.3) (for W incident with x) vanishes. Thus (4.3) is a genuine contour integral for fixed x, depending only on the homology class of the contour within $L_x \cap \mathcal{N}_{12}$. Simple direct calculation (using (3.2)) shows that (4.2) is indeed satisfied as required.

We could also envisage a more complicated cover $\mathcal{N}_1, \ldots, \mathcal{N}_r$ of $\mathcal{L}$, with Φ given a Čech description with respect to it by the family of holomorphic (homogeneous of degree -2) functions

(4.5) $$\Phi_{jk} = -\Phi_{kj} \quad \text{on } \mathcal{N}_{jk} = \mathcal{N}_j \cap \mathcal{N}_k$$

subject to

$$\Phi_{jk} + \Phi_{kl} + \Phi_{lj} = 0 \quad \text{on } \mathfrak{N}_{jkl} = \mathfrak{N}_j \cap \mathfrak{N}_k \cap \mathfrak{N}_l. \tag{4.6}$$

The expression for $\phi(x)$ is given, essentially as before, by (4.3), but now the contour Ξ is a branched one, consisting of various segments $\Xi_{jk} \subset \mathfrak{N}_{jk}$, over which Φ_{jk} is to be integrated, with endpoints $\mathbf{W}_{jkl} \in \mathfrak{N}_{jkl}$, and we sum over the contributions from these various Φ_{jk}. In fact (for appropriate $\mathfrak{Q}$) the general solution of (4.2) can be obtained in this way (cf. Eastwood, Penrose and Wells (1981), Penrose and Ward (1980)).

Similarly Maxwell's free-space equations can be solved in terms of elements

$$(\Theta, \Psi) \in H^1(\mathfrak{L}, \mathcal{O}(-4)) \oplus H^1(\mathfrak{L}, \mathcal{O}(0)). \tag{4.7}$$

The components of the Maxwell field tensor, in the (τ, ξ, η, ζ)-coordinate system, are given by

$$\begin{pmatrix} 0 & \theta_0 - \theta_2 + \psi_0 - \psi_2 & i(\theta_0 + \theta_2 - \psi_0 - \psi_2) & -2(\theta_1 + \psi_1) \\ \theta_2 - \theta_0 + \psi_2 - \psi_0 & 0 & 2i(\theta_1 - \psi_1) & -\theta_0 - \theta_2 - \psi_0 - \psi_2 \\ i(-\theta_0 - \theta_2 + \psi_0 + \psi_2) & 2i(-\theta_1 + \psi_1) & 0 & i(\theta_2 - \theta_0 - \psi_2 + \psi_0) \\ 2(\theta_1 + \psi_1) & \theta_0 + \theta_2 + \psi_0 + \psi_2 & i(\theta_0 - \theta_2 - \psi_0 + \psi_2) & 0 \end{pmatrix}$$

where

$$\begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{pmatrix} = \frac{1}{2\pi i} \oint_{\Xi} \begin{pmatrix} W_0^2 \\ W_0 W_1 \\ W_1^2 \end{pmatrix} \Theta(W) \Delta W, \qquad \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{2\pi i} \oint_{\Xi} \begin{pmatrix} \partial^2/\partial W_2^2 \\ \partial^2/\partial W_2 \partial W_3 \\ \partial^2/\partial W_3^2 \end{pmatrix} \Psi(W) \Delta W$$

(cf. Penrose (1969), (1977)), the notation $\Theta(W)$, $\Psi(W)$ indicating the appropriate $\Theta_{jk}(W)$, $\Psi_{jk}(W)$ to be integrated over Ξ and summed as before. The quantities $\theta_0, \theta_1, \theta_2$ define the anti-self-dual part of the Maxwell field and ψ_0, ψ_1, ψ_2 define the self-dual part. For a *real* Maxwell field we require

$$\theta_0 = \bar{\psi}_0, \qquad \theta_1 = \bar{\psi}_1, \qquad \theta_2 = \bar{\psi}_2,$$

at real points x (and, at complex points, $\theta_0(x) = \bar{\psi}_0(\bar{x})$, etc.) so we can use *either* $H^1(\mathfrak{L}, \mathcal{O}(-4))$ *or* $H^1(\mathfrak{L}, \mathcal{O}(0))$ to specify such a field, employing complex conjugation in place of using the other.

Recall that the Levi form signature for a smoothly bounded domain of holomorphy—i.e. natural (smooth) boundary of the domain of some holomorphic *function*—is always of the form $(+ \cdots +0 \cdots 0)$. But a function is a *zeroth* sheaf cohomology element (an H^0), whereas we are now more concerned with *first* cohomology elements (H^1's). There is an analogous concept to a domain of holomorphy for such objects, namely a smoothly bounded region $\mathfrak{D}$ on which is defined some holomorphic H^1 (i.e. an H^1 for a coherent analytic sheaf) which does not locally extend beyond the boundary $\partial\mathfrak{D}$. It follows from a theorem of Andreotti and Hill (1972) that, at each point of $\partial\mathfrak{D}$, $\mathfrak{D}$ either has precisely *one* degree of pseudoconcavity (i.e. Levi form signature of the form $(+ \cdots +0 \cdots 0\, -)$) or else is pseudoconvex with at least one zero in the Levi signature (i.e.

$(+\cdots+0\cdots00)$). Referring to the number of zeros in the Levi signature as the number, d, of *degrees of holomorphic flatness* we find, generally, that a smoothly bounded region on which is defined a locally inextendible holomorphic H^r has a number, q, of degrees of pseudoconcavity satisfying, at each point of its boundary,

$$r \geqslant q \geqslant r - d. \tag{4.8}$$

We have seen that for ordinary functions (i.e. H^0's), we could refer to the concept of CR-function on a CR-manifold $\mathcal{V}$, realized as a hypersurface in a complex manifold $\mathcal{C}$, and ask whether holomorphic extendibility on one side or the other is locally possible. The appropriate concept for H^r's can be stated in terms of the $\bar{\partial}_b$-cohomology of Andreotti & Hill (1972) and Hill & McKichan (1977); cf. also Polking & Wells (1976), Hill (1973). We find that an appropriate concept of a "CR-H^r" is *not* taken as the restriction to $\mathcal{V}$ of a holomorphic H^r in $\mathcal{C}$ but may be defined, rather, in terms of the $\hat{H}^r$ provided by an exact sequence

$$\cdots \to H^r(\mathcal{C}) \to \begin{matrix} H^r(\mathcal{C}^+) \\ \bigoplus \\ H^r(\mathcal{C}^-) \end{matrix} \to \hat{H}^r(\mathcal{V}) \to H^{r+1}(\mathcal{C}) \to \begin{matrix} H^{r+1}(\mathcal{C}^+) \\ \bigoplus \\ H^{r+1}(\mathcal{C}^-) \end{matrix} \to \cdots \tag{4.9}$$

where $\mathcal{V}$ separates $\mathcal{C}$ into the two pieces $\mathcal{C}^-$, $\mathcal{C}^+$ (so $\mathcal{C} = \mathcal{C}^- \cup \mathcal{C}^+ \cup \mathcal{V}$). Here $\hat{H}^r(\mathcal{V})$ denotes the required space of rth *hyperfunctional* $\bar{\partial}_b$-cohomology. (See Sato (1959–60), Kashiwara (1979).) The sheaf for each H is taken to be that of local cross-sections of some given holomorphic vector bundle over $\mathcal{C}$, and that for $\hat{H}$ to correspond in an appropriate sense to the restriction of this bundle to $\mathcal{V}$.

If we can arrange $H^{r+1}(\mathcal{C}) = 0$, which will be the case, for example, when $\mathcal{C}$ is *Stein* (i.e. a domain of holomorphy), then we obtain from (4.9)

$$\hat{H}^r(\mathcal{V}) = \frac{H^r(\mathcal{C}^-) \oplus H^r(\mathcal{C}^+)}{H^r(\mathcal{C})} \tag{4.10}$$

—and if $\mathcal{C}$ is Stein we can ignore the $H^r(\mathcal{C})$ $(= 0)$ when $r > 0$. We can loosely interpret the elements of $\hat{H}^r(\mathcal{V})$, in this description, as the space of possible "jumps" in H^r as we cross from one side of $\mathcal{V}$ to the other. In the particular case $r = n = 0$ (where $\mathcal{C}$ has complex dimension $n + 1$), with $\mathcal{V}$ being (a portion of) the real axis in $\mathbb{C}$, (4.10) provides the standard definition of 1-dimensional hyperfunctions on $\mathbb{R}$ (Sato (1959–60), Kashiwara (1979)). Hyperfunctions are generalizations of distributions and so they include, as special cases, ordinary C^∞ functions. The C^∞ CR-functions on $\mathcal{V}$ can thus be taken to be *particular* elements of $\hat{H}^0(\mathcal{V})$.

The concept of local (Lewy) extendibility of C^∞ CR-functions on $\mathcal{V}$, into regions of $\mathcal{C}$ locally bounded by $\mathcal{V}$ wherever there is at least one degree of pseudoconcavity, applies also to CR-hyperfunctions (cf. Hill and MacKichan (1977)). Moreover, there is a corresponding result for elements of $\hat{H}^r(\mathcal{V})$ generally: local holomorphic extendibility into the complex manifold on one side of $\mathcal{V}$ can always be achieved, in the case of a nondegenerate Levi form, whenever the number, q, of degrees of pseudoconcavity is not exactly r. More generally, if there are d degrees of holomorphic flatness, then local extendibility can always be achieved wherever

$$q > r \quad \text{or} \quad q + d < r, \tag{4.11}$$

which is the negation of (4.8).

Note that if both $\mathcal{C}^+$ and $\mathcal{C}^-$ have some pseudoconcavity everywhere along $\mathcal{V}$, as is the case when $\mathcal{V} = \mathbb{PT}_0$ or $\mathbb{T}_0$, then every CR-function (whether C^∞ or hyperfunctional) extends to both sides of $\mathcal{V}$ and is therefore always analytic. This shows that we cannot simply use CR-functions in a straightforward Čech approach to the CR-cohomology of such a $\mathcal{V}$, since nonanalytic elements of H^r will always occur for some $r > 0$. Indeed, when $\mathcal{V} = \mathbb{PT}_0$ or $\mathbb{T}_0$ we find that $\hat{H}^1(\mathcal{V})$ contains nonanalytic elements. In fact, solutions of the wave equation (4.2) or of Maxwell's equations on (regions in) *real* Minkowski space M can be represented by elements of an $\hat{H}^1(\mathcal{V})$ (respectively $\hat{H}^1(\mathbb{PT}_0, \mathcal{O}(-2))$ and $\hat{H}^1(\mathbb{PT}_0, \mathcal{O}(-4) \oplus \mathcal{O}(0)))$ and these fields certainly need not be analytic (cf. Wells (1981), Bailey, Ehrenpreis and Wells (1982)). Thus the direct Čech approach, with CR-functions on $\mathbb{PT}_0$, does not apply and a method such as using the sequence (4.9) or an explicit $\bar{\partial}_b$ approach is called for.[2]

This contrast between a direct Čech approach and such as the $\bar{\partial}_b$ method has its analogue in a related contrast between CR-structures which are realizable in terms of imbeddability in complex manifolds and those which are not. Let us consider two possible approaches to defining a complex manifold. In the first (method $\check{C}$), we consider the manifold to be built up from open patches on $\mathbb{C}^{n+1}$, a family of holomorphic transition functions F_{jk} being provided to "glue" these patches together, where the F_{jk} are subject to certain nonlinear relations closely analogous to (4.5) and (4.6). Likewise, the equivalence between two such complex manifolds so constructed is closely analogous to the condition for two Čech 1-cocycles Φ and Ψ to differ by a coboundary

$$\Phi_{jk} - \Psi_{jk} = \Lambda_k - \Lambda_j \quad \text{on } \mathcal{N}_{jk}$$

where Λ_j is holomorphic on $\mathcal{N}_j$. Here, the Λ_j are analogous to coordinate transformations on the individual patches. Finally, the concepts of taking refinements and direct limits are common both to Čech cohomology and to this method of complex-manifold building. Thus, method $\check{C}$ presents a complex manifold as a kind of nonlinear holomorphic H^1, presented according to a Čech prescription.

In the second method (method $\bar{\partial}$) the manifold is given first as a real manifold of dimension $2(n+1)$. An operator J (with $J^2 = -1$) is defined to act in the tangent spaces and to satisfy an integrability condition like (3.6) with (3.5). When phrased suitably, method $\bar{\partial}$ resembles a nonlinear version of the Dolbeault ($\bar{\partial}$) approach to the definition of a holomorphic H^1.

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[2] There are also other (essentially equivalent) methods of defining the concept of a (say C^∞) CR $- H^R$ on $\mathcal{V}$, which can provide the desired nonanalytic cohomology. One of these is to define the cohomology in terms of extensions of CR-vector-bundles over $\mathcal{V}$.

Because of the Newlander-Nirenberg theorem, these two methods of building a complex manifold are equivalent—and, likewise, these two approaches to constructing a holomorphic H^1 are equivalent. However, we have just seen that there is an inequivalence between the methods of defining the H^1 cohomology at the *boundaries* of complex manifolds. Such considerations led C. D. Hill to suggest, a number of years ago, that there ought to be a *difference between the $\check{C}$-method and $\bar{\partial}$-method of defining a complex manifold-with-boundary*, and that this difference should manifest itself when the manifold has one degree of pseudoconcavity at its boundary (or perhaps none, in cases where the Levi form is degenerate). The $\check{C}$-method would be as before, except that the patches would now have specified *boundary* portions (which would have to piece together suitably), given as (say) C^∞ real hypersurfaces in each coordinate patch. In the $\bar{\partial}$-method, the operator H would be required to attain C^∞-smooth boundary values at the boundary of the given $2(n+1)$-real-dimensional C^∞ manifold-with-boundary.

Both methods would provide the boundary $\mathcal{V} = \partial\mathcal{D}$, of the complex manifold $\mathcal{D}$ with a CR-structure, but with the $\check{C}$-method, $\mathcal{D}$ would always be locally extendible to the other side of $\mathcal{V}$ (simply by a slight extension of the relevant coordinate patch) whereas with the $\bar{\partial}$-method we would have no guarantee that this can be done. Indeed, the existence of locally nonextendible holomorphic H^1's at (a realizable) $\mathcal{V}$, when there is just one degree of pseudoconcavity, suggests also the existence of locally nonextendible complex manifolds-with-boundary with just one degree of pseudoconcavity. Moreover, with a Levi form of, say, signature $(+\,-)$, like that of a perturbed version $\mathcal{V} = \mathbb{P}\mathcal{T}_0$ of $\mathbb{PT}_0$, as described in §3, we may expect that the CR-manifold $\mathcal{V}$ can sometimes be realizable as a boundary *neither* on one side *nor* on the other.

One approach to the construction of such nonrealizable CR-structures is to produce $\mathbb{P}\mathcal{T}_0$, as in §3, where the space-time $\mathcal{M}$ and hypersurface $\mathcal{U}$ are suitably nonanalytic. However the mere nonanalyticity of the resulting CR-structure is not sufficient to ensure its nonrealizability. The reason is, of course, that the real hypersurface defining a realizable CR-manifold need not itself be analytic. Thus, LeBrun (1980) was forced into some more subtle (nonlocal) considerations in order to prove his nonrealizability result for this case.

The related but slightly different earlier suggestion due to Sparling was to use, instead, the Ward (1979) "twisted photon" construction. According to this construction (a special case of Ward's (1977) twistor method of generating self-dual Yang-Mills fields), self-dual Maxwell fields are represented by holomorphic line bundles over the appropriate regions of $\mathbb{PT}.$, rather than by elements $\Psi \in H^1(\mathcal{L}, \mathcal{O}(0))$, as in (4.7). The bundles are constructed using the multiplicative $\mathcal{O}_*$ (zeros excluded), rather than the additive $\mathcal{O}(0)$, the relation between the two being achieved via the familiar exact sequence

$$(4.12) \qquad 0 \to \mathbb{Z} \xrightarrow{\times 2\pi i} \mathcal{O}(0) \xrightarrow{\exp} \mathcal{O}_* \to 0.$$

Now, real Maxwell fields on (open) regions $\mathcal{R}$ of real Minkowski space M are equivalent to self-dual Maxwell fields on $\mathcal{R}$ (the relevant mutually inverse maps being "take the self-dual part" and "take twice the real part"). To express such fields we can thus either use elements of

$$\hat{H}^1(\mathcal{K}, \mathcal{O}(0)) \tag{4.13}$$

where $\mathcal{K}$ is the region of $\mathbb{PT}_0$ swept out by the projective lines representing the points of $\mathcal{R}$ (as in (4.9)), or else use the corresponding line bundles over $\mathcal{K}$. Such real Maxwell fields certainly need not be analytic, and may even be hyperfunctional. Let us take the field to be C^∞ but nowhere analytic. It may then be represented by a C^∞ CR-line bundle over $\mathcal{K}$ in accordance with the exponentiation procedure of (4.12) as applied to the element of (4.13) representing the field. This bundle is not extendible to a holomorphic line bundle over some open neighbourhood of $\mathcal{K}$ in $\mathbb{PT}$., for if it were, the self-dual Maxwell field on the corresponding open region of $\mathbb{C}M$ that it represents would restrict down to (the self-dual part of) the nonanalytic field that we started with, which is a contradiction since Maxwell fields on open regions of $\mathbb{C}M$ are necessarily holomorphic (and therefore real-analytic). By choosing our original self-dual field to be either of positive or negative frequency, we can ensure *one*-sided extendibility of the resulting bundle into $\mathbb{PT}_-$ or $\mathbb{PT}_+$, respectively (cf. Wells (1981), Bailey, Ehrenpreis and Wells (1982)).

By a slight extension of Sparling's above line of argument, one can translate the above statements concerning inextendible or one-side-extendible CR-line bundles over $\mathcal{K}$ to nonrealizable or one-side-realizable CR-manifolds. The CR-line bundle over $\mathcal{K}$ is itself a CR-manifold with $(+0-)$ Levi signature (and the one direction of holomorphic flatness defines the fibre direction). Now the structure that a holomorphic line bundle $\mathcal{B}$ possesses, over and above that of its complex-manifold structure, can be represented in terms of the holomorphic vector field

$$\lambda\frac{\partial}{\partial\lambda} \tag{4.14}$$

on $\mathcal{B}$, where λ is a fibre coordinate on each fibre. Likewise, the bundle structure of the above CR-line bundle over $\mathcal{K}$—which we now call $\mathcal{V}$—is also characterized by (4.14), which is now a CR-vector field on $\mathcal{V}$. Suppose that $\mathcal{V}$ forms a boundary to a complex manifold-with-boundary $\mathcal{D}$. It follows from the Lewy extension property (and the $(+0-)$ Levi signature) that the vector field (4.14) extends locally into $\mathcal{D}$ and defines $\mathcal{D}$ (locally near $\mathcal{V}$) as a holomorphic line bundle over some open neighbourhood in $\mathbb{PT}_+$ or $\mathbb{PT}_-$ of $\mathcal{K}$. (Though the Lewy extension property refers, in the first instance, only to CR-functions and not to vector fields, the extension to CR-vector fields is easy to achieve by various means, e.g. take components in local CR-coordinates.) It follows from the above arguments (essentially) that our inextendible, or one-side-extendible CR-line bundles are neither-side-realizable or one-side-realizable CR-manifolds, respectively. In the latter case, we have examples of Hill's complex manifolds-with-boundary of the $\bar\partial$-type which are not realizable by the $\check{C}$-method, since they are not extendible beyond their C^∞ smooth boundaries.

The above discussion has been given partly in outline. It depends to some extent upon the details of twistor geometry and the twistor constructions of physical fields. Not all of this is necessary, however, for the construction of neither-side-realizable and one-side-realizable CR-manifolds. A simplified example due to C. D. Hill, G. A. J. Sparling and the author can be given, where $\mathcal{V} = \partial\mathcal{D}$, with real dimension 5 and Levi signature $(0\,-)$, where the complex 3-manifold-with-boundary $\mathcal{D}$ cannot be extended as a complex manifold beyond its C^∞ boundary $\mathcal{V}$. This example has an advantage over what has been presented above in that the extendibility emerges as manifestly a *local* rather than a global obstruction. (The relation between cohomology classes on portions of $\mathbb{PT}$ and physical fields depends upon a nonlocal correspondence.) A fully detailed argument will be presented elsewhere, but the gist of the construction is as follows.

Let $\mathbb{S}$ be the unit 3-sphere in $\mathbb{C}^2$, and define $\mathcal{K}$ and $\mathfrak{E}$ to be the intersections with $\mathbb{S}$ and with the closed exterior of $\mathbb{S}$, respectively, of the open ball of radius ε (< 2) centered about some point of $\mathbb{S}$. Then $\mathfrak{E}$ is a complex 2-manifold-with-boundary, with $\mathcal{K} = \partial\mathfrak{E}$ and Levi signature $(-)$. Choose an element $\Phi \in H^1(\mathfrak{E} - \mathcal{K}, \mathcal{O})$ which has C^∞ boundary values on $\mathcal{K}$ but which is not locally extendible beyond $\mathcal{K}$. By the aforementioned standard exponentiation process, construct a holomorphic line bundle over $\mathfrak{E} - \mathcal{K}$ which joins smoothly to a CR-line bundle over $\mathcal{K}$. These two pieces of line bundle together provide the required complex 3-manifold-with-boundary $\mathcal{D}$ (as its total space) and the portion over $\mathcal{K}$ provides the required C^∞ CR-5-manifold $\mathcal{V}$, of Levi signature $(0\,-)$, beyond which $\mathcal{D}$ is not locally extendible as a complex manifold. For the proof we appeal to the same argument as before, which uses Lewy extension of the vector field (4.14) to show that any extension of $\mathcal{D}$ across $\mathcal{V}$ as a complex manifold is also (locally) an extension of $\mathcal{D}$ as a holomorphic vector bundle. But such extension would imply that Φ extends across $\mathcal{K}$, which is a contradiction.

The Hill philosophy[3] would suggest that a corresponding example should exist in which $\mathcal{D}$ is a 2-complex-dimensional with Levi signature $(-)$ at its C^∞ boundary $\mathcal{V}$. Perhaps the original Nirenberg example bounds on one side and thus provides such a $\mathcal{V}$?

It is remarkable, and perhaps even somewhat ironic that such seemingly esoteric matters as nonrealizable CR-structures and the delicate dividing line between C^ω and C^∞ should have significant connections with physics. But it seems to be so. (The infinite-dimensionality of $H^1(\mathbb{PT}_+, \mathcal{O}(r))$—and hence of physical massless fields—can be attributed to Lewy nonextendibility across $\mathbb{PT}1$.) What would Poincaré have made of all this?

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[3]Recently a large class of nonrealizable CR-manifolds has been presented by Jacobowitz and Trèves (1982). Also Kuranishi (1982) has shown that in dimension $2n + 1 > 7$ and for positive-definite Levi form all CR-manifolds are realizable. All these results are consistent with the Hill philosophy.

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