The disadvantage of the compromise, especially in the case of [**R**], is that the supporting text is extremely condensed and unmotivated—hence really inaccessible. A teacher who wished to present a typical entry would require a year of preparation to plan the introduction of preparatory material. This is compounded in [**R**] by the lack of references. As one example: "A Fréchet space which is not distinguished", the excellent index tells us what the words mean. The construction is self-contained and very difficult with no reference to author or other source. This is the last item in [**4**] where also a reference is given—it would have been better for this information to be in [**R**]. (The easier construction of a nondistinguished l.c. space is in [**9**].) The reviewer is listed twice on p. 65 in the disguise of the letter $W$. Alas his chance for immortality in connection with $W$-barrelled spaces has been annulled by Steve Saxon's result (not in [**R**]) that $W$-barrelled is equivalent to second category [**9**, #5-2-301].

The index should be emended: Echelon space 61; Normal topology 51.

## References


R. The subject of this review.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

0273-0979/82/0000-0977/$01.75

*Convexity theory and its applications in functional analysis*, by L. Asimow and A. J. Ellis, Academic Press, London, v + 266 pp., $56.00. ISBN 0-1206-5340-0

This book focuses on the role of compact convex sets in functional analysis. We will begin this review by trying to indicate why this role has been an important one and by giving a brief description of the historical evolution of research in this area. We will then turn to commenting directly on the contents and contribution of the book under review.

One reason for the central role of compact convex sets in functional analysis is their ubiquity—as evidenced by the Banach-Alaoglu theorem that the unit ball of the dual space of a Banach space is weak*-compact. Compact convex sets play a key role, for example, in the fields of function algebras, group representations, and $C^*$-algebras. If $A$ is a function algebra, the state space $S(A)$ is defined to be the set of linear functionals in the dual unit ball with value 1 at the identity; this compact convex set contains all multiplicative linear functionals and thus contains a homeomorphic copy of the spectrum of $A$. If $G$ is a locally compact group, the set of continuous positive definite functions with value at most 1 at the identity is compact and convex. There is a construction which for each such function associates a representation of $G$; all strongly continuous unitary representations of $G$ arise in this way. For another example, if $A$ is a unital $C^*$-algebra, the states of $A$ (i.e. the positive functionals with value 1 at the identity) form a compact convex set, and again states of $A$ yield representations of $A$.

For a final example, let $X$ be a compact Hausdorff space, $C(X)$ the Banach space of continuous real-valued functions on $X$, and $M$ a closed subspace of $C(X)$, separating points of $X$ and containing the constant functions. Let $S(M)$ be the set of functionals in the dual unit ball of $M$ which take the value 1 at 1. ($S(M)$ is called the state space of $M$.) Then $S(M)$ is convex and compact in the weak* topology and completely determines $M$. In fact, the evaluation map is an isometry (and an order isomorphism) from $M$ onto the space $A(S(M))$ of continuous affine functions on $S(M)$ (where we give $A(S(M))$ the supremum norm and pointwise ordering). This example is universal in the following sense. If $K$ is any compact convex set in a locally convex topological vector space, then the evaluation map is an affine homeomorphism of $K$ onto the state space of $A(K)$ (the ordered Banach space of continuous affine functions on $K$). This duality of $K$ and $A(K)$ was discovered by Kadison and has come to be the standard context for working with compact convex sets. In general if $K$ is any compact convex set, we will refer to the elements of $K$ as states.

The preceding examples illustrate the kinds of ways in which compact convex sets arise in functional analysis. For any convex set $K$, a convex combination $x = ty + (1 - t)z$ can be thought of as expressing $x$ as a "mixture" of $y$ and $z$. Points that cannot be expressed as such a mixture (for $y$, $z$ distinct and $t \neq 0, 1$) are the extreme points ("pure states") of $K$. As one might expect, they play a key role in analyzing the structure of $K$ and in applications. The starting point here is, of course, the Krein-Milman theorem: every compact convex set (in a locally convex space) is the closure of the convex hull of its extreme points. In the case of locally compact groups, the extreme points of the compact convex set of positive definite functions described above are precisely those which give rise to *irreducible* representations. This was used by Gelfand and Raikov to establish the existence of a complete set of irreducible representations. In the context of $C^*$-algebras, again pure states give rise to irreducible representations; Segal used this to establish the existence of a complete set of irreducible representations for $C^*$-algebras. (This construction has become known as the G-N-S construction.) Finally, in the function algebra context, each pure state is multiplicative, but not all multiplicative linear functionals are pure states. The closure of the pure states can be identified with the Shilov boundary of the algebra—the minimal closed subset of the maximal ideal space on which each function in the algebra achieves its maximum.

Given the existence of many pure states, it is natural to pursue the question of decomposition of arbitrary states into mixtures of pure states. In finite dimensions, Minkowski's theorem asserts that every state is a convex combination of pure states. The Krein-Milman theorem gives one generalization: every state is a limit of convex combinations of pure states. It seems reasonable to hope each state could be expressed as an *integral* of pure states. One way to make this precise is to recall that convergence in $K$ is equivalent to pointwise convergence on $A(K)$. Thus if $\mu$ is a measure on $K$ and $x$ is any state in $K$, then we write

$$x = \int_K k\, d\mu(k) \quad \text{if } a(x) = \int_K a(k)\, d\mu(k)$$

for all $a$ in $A(K)$. (If this holds, $x$ is said to be the barycenter of $\mu$.) Then the question becomes whether the measure $\mu$ can be taken to live on the set $\partial_e K$ of pure states of $K$, i.e. $\mu(\partial_e K) = 1$. Choquet showed that this can always be done if $K$ is metrizable. (If $K$ is not metrizable, so that $\partial_e K$ may not even be measurable, then the Choquet-Bishop-de Leeuw theorem states that $\mu$ can be chosen to be a boundary measure; this reduces to the previous result if $K$ is metrizable.) As one might expect, these decompositions for states can be used to establish decompositions of representations into irreducible representations for locally compact groups and $C^*$-algebras.

What about the uniqueness of this decomposition of states into pure states? In finite dimensions, each point is a unique convex combination of pure states iff the convex set is a simplex. Choquet gave the definition for an arbitrary compact convex set being a simplex. (The intersection of homothets of $K$ should again be a homothet of $K$; equivalently $A(K)^*$ should be a vector lattice for the positive cone generated by $K$.) He then showed that these compact convex sets were precisely those with the desired uniqueness property.

This general theory has come to be known as Choquet theory; it dates from the late fifties. The theory and some of its applications are dealt with in R. Phelps' book *Lectures on Choquet's theorem* and in E. Alfsen's *Compact convex sets and boundary integrals*. Choquet theory is the main topic of Phelps' book and the subject of almost half of Alfsen's book. Given the excellent presentation of Choquet theory in these two books, the authors of the book under review wisely decided that another presentation in depth was not needed. Instead, they very quickly derive the basic results of Choquet theory, and use this as a tool throughout the book. The main focus of the book is instead the developments in the field in recent years.

The direction of most work regarding compact convex sets in the last twenty years (and the primary subject of this book) has been the use of $K$ as a dual object in exploring the structure of $A(K)$. As indicated above, $K$ determines the order and norm on $A(K)$, and so it is not surprising that related properties for $A(K)$ are reflected in corresponding properties of $K$. For example, $A(K)$ will be a vector lattice iff $K$ is a simplex whose set of pure states is closed (a "Bauer simplex"). However, it often turns out that if $K$ arises as the state space of an algebra, then this algebra structure on $A(K)$ is reflected in the geometry of $K$. For example, if $A = C(X)$, then isomorphisms of $A$ are in 1-1 correspondence with affine homeomorphisms of $S(A)$. (In fact, the multiplication on $A = A(S(C(X)))$ can be recovered by noting that $fg = h$ iff $f(k)g(k) = g(k)$

for all pure states $k$.) More generally, Kadison showed that affine homeomorphisms of the state space of a unital $C^*$-algebra induce Jordan isomorphisms of the $C^*$-algebra. (A Jordan isomorphism is a linear map preserving the symmetrized product $ab + ba$.) (In this case, however, recovering the Jordan product from the state space is much more subtle than for $C(X)$.)

Thus for $C^*$-algebras such algebraic notions as the center, one-and two-sided ideals, the primitive ideal space, etc. have been found to have characterizations in terms of the state space. For function algebras, the state space has been used fruitfully in investigating the notions of peak sets and interpolation sets.

A related direction has been the study of certain classes of compact convex sets and the related $A(K)$ spaces. For example, much work has been devoted to the study of simplex spaces (spaces $A(K)$ for $K$ a simplex) and the question of characterizing the state spaces of $C^*$-algebras.

The book under review focuses on these relationships between the structure of $K$ and that of $A(K)$. We will now summarize the chapters of the book. (It should be said that research in recent years has gone in a variety of directions, quite a few of which are discussed here, and we will not attempt to present a thorough summary.)

The first chapter ("Preliminaries") gives a speedy development of prerequisites (with the assumption that the reader has taken at least a basic course in functional analysis). The approach here focuses on the tools of the polar calculus and gauges (Minkowski functionals of convex sets containing zero), and leads up to the Choquet-Bishop-de Leeuw theorem and the Bishop-Phelps theorem.

The second chapter ("Duality in Ordered Banach Spaces") includes a discussion of Choquet simplexes, of abstract $L$ and $M$ spaces, and of the Riesz decomposition property, as well as the basic facts on the $K$-$A(K)$ framework. The authors often have chosen to state results in a more general context than Alfsen and Phelps (who stick primarily to spaces of the form $A(K)$).

The third chapter deals with simplex spaces. There are many beautiful results here. Several characterizations of simplexes among compact convex sets are given. (For example, a compact convex set is a simplex iff each closed face is split iff each closed face is parallel.) The Poulsen simplex is discussed in detail. (This amazing simplex $K$ has the property that $\partial_e K$ is dense in $K$, and it is uniquely determined by this property among metrizable simplexes. Furthermore, if $K$ is any compact metrizable simplex, then $K$ is affinely homeomophic to a closed face of the Poulsen simplex.) The authors also give Haydon's proof of the result of Choquet that a topological space $X$ is homeomorphic to $\partial_e K$ for $K$ a compact metrizable simplex iff $X$ is a Polish space (i.e. a separable complete metrizable space). In this chapter, as throughout the book, the authors give very nice constructions to serve as counterexamples to natural conjectures and to illuminate the spirit of the theorems.

The fourth chapter ("Complex Function Spaces") is the primary chapter that works in the context of complex scalars. Included here are the complex versions of results on uniqueness of representing measures and on Lindenstrauss spaces (Banach spaces whose dual is isometric to a space $L'(\mu)$). However, the main thrust of the chapter is the study of function algebras. In general a function algebra $A$ may contain almost no real valued functions, and so will not contain $A(K)$ where $K$ is the state space of $A$. Nevertheless, $S(A)$ proves to be useful in analyzing facts about $A$. The primary tool of the authors is the use of the *complex state space*, the convex hull of $\alpha S(A)$ and $\beta S(A)$ for suitable complex scalars $\alpha$ and $\beta$. The authors use this framework to derive many of the classical results of peark and interpolation sets for function algebras. (The Hoffman-Wermer theorem, the Bishop peak point theorem, and the Rudin-Carleson theorem appear as corollaries to the developments here.)

The final chapter discusses convexity theory for $C^*$-algebras. The center and primitive ideal space of a $C^*$-algebra are given geometric characterizations, and the Dauns-Hoffman theorem (that the center consists of those elements of $A$ which induce a continuous map on the primitive ideal space) is proved. The results of Effros and Prosser giving a duality of ideals of $A$ and faces of $S(A)$ are derived. The book ends with a discussion of the results of Alfsen, Hanche-Olsen, Størmer, and Shultz characterizing those compact convex sets which are affinely homeomorphic to the state spaces of Jordan and $C^*$-algebras. The authors do an excellent job of giving a self-contained summary of these characterizations, including proofs of several key results.

The authors have done a nice job of presenting a large amount of diverse material in a cohesive and self-contained fashion. The material is presented clearly and succinctly, with well-chosen examples. The choice of topics is excellent—a variety of the most appealing results of the last twenty years or so in this area. (This is also an area in which the authors themselves have been quite active.)

The book would perhaps be a little difficult to use for quick reference. Results are sometimes stated in fairly complicated notation, with the notation (and some assumptions) explained in the preceding text. Thus a reader who is only browsing may have some difficulty following the spirit of the lemmas. (This should not cause any problems for the serious reader; in any case, this is only a localized problem.) In general the book makes for very enjoyable reading.

The material here has little overlap with that available elsewhere in books (e.g. the books of Alfsen and Phelps). I highly recommend the book for the functional analyst interested in a self-contained presentation of many of the most interesting results in this field in recent years.

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BULLETIN (New Series) OF THE
AMERICAL MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

0273-0979/82/0000-1160/$01.75

*John von Neumann and Norbert Wiener, from mathematics to the technologies of life and death*, by Steve J. Heims, MIT Press, Cambridge, Mass., 1980, xviii + 547 pp., $19.95 (HB), $10.95 (PB). ISBN 0-2620-8105-9

By its title this book alerts the reader without any circumlocution that the author is not concerned primarily with writing biography but has set out to compose a contemporary morality play. His symbolic protagonists are Saint