ARITHMETIC GROUPS
ACTING ON COMPACT MANIFOLDS

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In this note we announce results concerning the volume preserving actions of arithmetic subgroups of higher rank semisimple groups on compact manifolds. Our results can be considered as the first rigidity results for homomorphisms of these groups into diffeomorphism groups and show a sharp contrast between the behavior of actions of these groups and actions of free groups.

Let G be a connected semisimple Lie group with finite center, such that every simple factor of G has R-rank ≥ 2. Let $\Gamma \subset G$ be a lattice. Then Γ is known to be finitely generated and arithmetic (the latter being a result of Margulis). If Γ is cocompact, then by standard arithmetic constructions one may have homomorphisms $\Gamma \to K$ where K is a compact Lie group and the image of Γ is dense. Thus Γ acts isometrically (and ergodically) on the homogeneous spaces of K. Our first result concerns perturbations of isometric actions. We recall that if one has an isometric diffeomorphism of a Riemannian manifold, then a perturbation of this diffeomorphism is not likely to be isometric. In fact hyperbolicity rather than isometry is typical of properties of a diffeomorphism that are preserved under a perturbation. The same remarks obviously apply as well to actions of free groups. However, with Γ as above we have the following.

THEOREM A. *Let M be a compact Riemannian manifold, $\dim M = n$. Set $r = n^2 + n + 1$. Assume Γ acts by smooth isometries of M. Let $\Gamma_0 \subset \Gamma$ be a finite generating set. Then any volume preserving action of Γ on M which*

(i) *for elements of Γ_0 is a sufficiently small C^r-perturbation of the original action, and*

(ii) *is ergodic,*
actually leaves a C^0-Riemannian metric invariant. In particular, there is a Γ-invariant topological distance function and the action is topologically conjugate to an action of Γ on a homogeneous space of a compact Lie group K defined via a dense range homomorphism of Γ into K.

We conjecture that without the ergodicity assumption one can still deduce the existence of a Γ-invariant C^0-Riemannian metric. The next result, proved by similar methods (together with results of Margulis [2] and Raghunathan [3]), is in the direction of the following conjecture.

CONJECTURE. Let G, Γ be as above. Let $d(G)$ be the minimal dimension of a nontrivial real representation of the Lie algebra of G, and $n(G)$ be the minimal dimension of a simple factor of G. Let M be a compact manifold, $\dim M = n > 0$. Assume (i) $n < d(G)$, and (ii) $n(n+1) < 2n(G)$. Then every volume preserving action of Γ on M is an action by a finite quotient of Γ. In particular, there are no volume preserving ergodic actions of Γ on M.

With the additional assumption of "near C^r-isometry" on the generators we can verify the final assertion of this conjecture. More precisely, if ξ is a metric on a compact manifold, $\mathcal{O}$ is a C^r-neighborhood of ξ in the space of (smooth) metrics with the same volume density as ξ, and S is a set of diffeomorphisms, we say that ξ is $(\mathcal{O}, S)$-invariant if $f^*\xi \in \mathcal{O}$ for all $f \in S$.

THEOREM B. *Let G, Γ be as above and M a compact manifold, $\dim M = n$. Assume $n(n+1) < 2n(G)$. Set $r = n^2 + n + 1$. Let $\Gamma_0 \subset \Gamma$ be a finite generating set. Then for any smooth Riemannian metric ξ on M, there is a C^r-neighborhood $\mathcal{O}$ of ξ such that there are no volume preserving ergodic actions of Γ on M for which ξ is $(\mathcal{O}, \Gamma_0)$-invariant.*

A similar result is true (and one puts forth a similar conjecture) for lattices $\Gamma \subset G_k$ where k is a totally disconnected local field of characteristic 0, but now without any restriction on the dimension of M. Here G is a connected algebraic k-group, almost k-simple, with k-rank$(G) \geq 2$.

We now give a very broad indication of some aspects of the proof of Theorem A. The first step is to apply the superrigidity theorem for cocycles proved in [4, 5, 6], a generalization of Margulis' superrigidity theorem. When combined with some general structural properties of algebraic groups, Kazhdan's property, ergodicity, and the Furstenberg-Kesten theorem on asymptotic behavior of the products of random matrices [1], the superrigidity theorem for cocycles yields conditions under which an action on a vector bundle will have a measurable invariant metric. The problem then is to convert this into the existence of a C^0 invariant metric. Kazhdan's property, ergodicity, and "near isometry" of Γ_0 enables one to give L^2 estimates relating a measurable invariant metric to an arbitrary smooth metric. Rather than only applying these results directly to the tangent bundle of M, however, we also need to apply them to the action on certain jet bundles defined on the (special) frame bundle of M. (The frame bundle is of infinite volume so the results do not apply directly, but under our hypotheses ergodic components of the action of Γ on the frame bundle will have finite measure.) With these measurable invariant metrics on jets over the frame bundle, we construct a Sobolev type space in which Γ will act unitarily. We then show that Kazhdan's property yields a fixed vector in this space and via our L^2 estimates and the usual Sobolev embedding theorem, this yields a continuous Γ-invariant function on the (special) frame bundle. We use this function combined with some supplementary ergodic theoretic arguments to construct the invariant C^0 metric.

Full details, further results, and extensions of the above conjectures will appear in [7].

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