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DIAGONALIZING MATRICES OVER OPERATOR ALGEBRAS

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1. Introduction. Let A_0 be a C^*-algebra and A be the algebra of $n \times n$ matrices with entries in A_0. If A_0 acting on a (complex) Hilbert space H_0 is a faithful representation of A_0, then A acting as matrices on the n-fold direct sum H of H_0 with itself is a faithful representation of A. As a subalgebra of $B(H)$, the algebra of all bounded operators on H, A acquires an adjoint and norm structure relative to which it is a C^*-algebra. This structure can be described independently of the representations—in particular, the operator in $B(H)$ adjoint to (a_{jk}) is the element of A whose matrix has a_{kj}^* as its j,k entry. If A_0 is the (algebra of) complex numbers C, then A is the algebra of $n \times n$ complex matrices and each normal element a can be "diagonalized"—that is, there is a unitary element u in A such that uau^{-1} has all its nonzero entries on the diagonal.

With A_0 a general C^*-algebra, can each normal element of A be diagonalized? In §2, we give a construction (based on homotopy groups of spheres) to show that this (general) question has a negative answer. The main result is discussed in §3. If A_0 is a von Neumann algebra, diagonalization of normal operators is always possible. More generally,

THEOREM. *If R_0 is a von Neumann algebra, R is the algebra of $n \times n$ matrices over R_0, and S is a commutative subset of R with the property that a^* is in S if a is in S, then there is a unitary element u in R such that uau^{-1} has all its nonzero entries on the diagonal for each a in S.*

2. An example. Let A_0 be the algebra $C(S^4)$ of continuous complex-valued functions on the 4-sphere S^4 and let A be the algebra of 2×2 matrices with entries in A_0. View S^3 as the unit sphere in two-dimensional Hilbert space C^2 and consider the standard action of $SU(2)$ (the group of 2×2 unitary matrices of determinant 1) on C^2. The mapping that takes u in $SU(2)$ to the vector $u(1,0)$ is a homeomorphism of $SU(2)$ onto S^3. From **[2]**, $\pi_4(S^3)$ is the additive group of integers modulo 2. Let u_0 be an essential mapping of S^4 into $SU(2)$ (that is, into S^3). The algebra A can be viewed as continuous mappings of S^4 into $B(C^2)$. Thus u_0 is a unitary (hence normal) element of A. Suppose u is a unitary element of A that diagonalizes u_0. Then $u(p)u_0(p)u(p)^{-1}$ is a 2×2 diagonal matrix over C for each p in S^4. Let $\theta(p)$ be the complex conjugate of the determinant of $u(p)$, let $u_1(p)$ be $\left[\begin{smallmatrix}\theta(p) & 0\\ 0 & 1\end{smallmatrix}\right]$, and let $v(p)$ be $u_1(p)u(p)$.

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Then $v(p)$ is in $SU(2)$, $v(p)u_0(p)v(p)^{-1} = u(p)u_0(p)u(p)^{-1}$, and v is a unitary element in A. Let f and g be two continuous mappings of S^4 into $SU(2)$ that take a "base point" p_0 in S^4 onto (the base point) I in $SU(2)$. Let fg denote the mapping that assigns to p in S^4 the group product $f(p)g(p)$ in $SU(2)$. Let $\{f\}$, $\{g\}$, and $\{fg\}$ be the corresponding elements (homotopy classes) in $\pi_4(SU(2))\,(=\pi_4(S^3))$. From [**1**], $\{f\}\{g\} = \{fg\}$. Moreover $\pi_4(SU(2))$ is abelian. Thus

$$\{vu_0v^{-1}\} = \{v\}\{u_0\}\{v^{-1}\} = \{u_0\}\{v\}\{v^{-1}\} = \{u_0\}\{vv^{-1}\} = \{u_0\} \neq 0.$$

But $v(p)u_0(p)v(p)^{-1}$ is diagonal and in $SU(2)$ and hence has the form $\begin{bmatrix} \lambda(p) & 0 \\ 0 & \overline{\lambda(p)} \end{bmatrix}$, where $|\lambda(p)| = 1$. Thus vu_0v^{-1} maps S^4 into a subset of $SU(2)$ homeomorphic to S^1 and $\{vu_0v^{-1}\} = 0$—a contradiction. Hence u_0 cannot be diagonalized.

3. Matrices over von Neumann algebras. Let R_0, R, and S be as in the theorem of §1. Let e_j be the element in R whose only nonzero entry is the identity at the j,j position. Then $e_1,\ldots,e_n$ are n orthogonal equivalent projections in R with sum the identity element of R. Suppose we can find n orthogonal equivalent projections $f_1,\ldots,f_n$ in R with sum the identity element such that each f_j commutes with every element of S. From various results in the comparison theory of projections in von Neumann algebras, we can conclude that e_j and f_j are equivalent in R for j in $\{1,\ldots,n\}$. Let v_j be a partial isometry in R with initial projection f_j and final projection e_j. Then $\sum_{j=1}^{n} v_j$ is a unitary element u in R such that $uf_ju^{-1} = e_j$ for j in $\{1,\ldots,n\}$. Since f_j commutes with each a in S (by assumption), uau^{-1} commutes with $e_j\,(=uf_ju^{-1})$ for each j in $\{1,\ldots,n\}$. Hence uau^{-1} is diagonal for each a in S.

The problem then is: Can we find $f_1,\ldots,f_n$ with the properties described? Does the "relative commutant" of S in R contain n orthogonal equivalent projections with sum the identity? We have little control over this relative commutant. From Zorn's lemma, S is contained in some maximal abelian (selfadjoint) subalgebra A of R. Each such A is contained in the relative commutant. But S may itself be such an A, in which case, the relative commutant is a maximal abelian subalgebra of R. Thus we must be prepared to (and it suffices to) find $f_1,\ldots,f_n$ as described in an arbitrary maximal abelian subalgebra of R. In effect, we must develop a comparison theory of projections in a maximal abelian subalgebra of R relative to R. The last of a series of results leading to such a theory is

THEOREM. *If R is a von Neumann algebra and each type I_k central summand of R is such that k is divisible by n, then each maximal abelian subalgebra of R contains n orthogonal equivalent projections with sum the identity element of R. In particular, this is true of the von Neumann algebra of $n\times n$ matrices over a von Neumann algebra.*

The full account of these results deals with the case where R_0 is countably decomposable to avoid complicated but peripheral higher cardinality considerations.

4. Related questions. There are a number of other avenues of study indicated by the foregoing discussion and results. We mention a few. For which compact Hausdorff spaces X is diagonalization of normal matrices over $C(X)$ possible in general? For 2×2 matrices? For 3×3 matrices? What is the relation between "n-diagonalizability" and "m-diagonalizability"? Certain types of normal elements may be diagonalizable in all circumstances—which are they? What "relative comparison theory" is possible for other von Neumann subalgebras of von Neumann algebras? For C^*-subalgebras of a von Neumann algebra?

BIBLIOGRAPHY

1. B. Eckmann, *Über die Homotopiegruppen von Gruppenräumen*, Comment. Math. Helv. **14** (1942), 234–256.
2. H. Freudenthal, *Über die Klassen der Sphärenabbildungen*, Compositio Math. **5** (1937), 299–314.

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