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BOOK REVIEWS

Fuzzy sets and systems: Theory and applications, by Didier Dubois and Henri Prade, Mathematics in Science and Engineering, vol. 144, Academic Press, New York, 1980, xvii + 393 pp., $49.50.

This book effectively constitutes a detailed annotated bibliography in quasi-textbook style of the some thousand contributions deemed by Messrs. Dubois and Prade to belong to the area of fuzzy set theory and its applications or interactions in a wide spectrum of scientific disciplines. The individual with an existing research commitment in this area will find the book competently written and an invaluable time-saver in developing an awareness of existing literature in English, French or German although the emphasis reflects, heavily at times, the authors' favorite topics. The mathematician wishing to precipitate out the mathematical and conceptual core of fuzzy set theory will find it frustrating reading, however. As I see it, the difficulty lies with how the subject is defined. Fuzzy set theory is not well delineated mathematically. It is determined by a set of papers which either have the word 'fuzzy' in the title or are authored by someone who has written such a paper. I am somewhat horrified that the authors include a bibliography specifically entitled 'nonfuzzy literature'. The dangerous insularity of the field has been noted in [**1**, **17**].

It seems justifiable, then, to direct this review in large part to fuzzy set theory generally.

1. What passes for a theory. This section briefly overviews the main aspects of the 'mathematical core' of fuzzy set theory as represented by the first two chapters of "Part II: Mathematical Tools" (such unqualified references are to the book under review). For lack of major theorems and juicy open questions, I do not feel this material has yet coagulated into what mathematicians would call a theory.

By identifying subsets of a set with their characteristic functions, the Boolean algebra structure of the set of subsets of a set derives, via pointwise operations, from the Boolean algebra structure of the two-element set $2 = \{0, 1\}$, 0 for 'false' and 1 for 'true'. Fuzzy set theory generalizes 2^X to $[0, 1]^X$ where $[0, 1]$ is the unit interval. Elements of $[0, 1]^X$ are called *fuzzy sets* (*with universe* X). The rigorization of probability theory by Kolmogoroff developed from a frequency paradigm. As discussed in Chapter 1 of Part IV, fuzzy sets have a more 'subjective' paradigm. (As a mathematician, I found this explicit foundational link to human psychology disquieting; perhaps the authors did too, since the placement of this discussion is far from the beginning of the book.) As a result, there is little intuitive basis to decide which operations $[0, 1]^n \to [0, 1]$ should generalize the Boolean operations $2^n \to 2$. The authors avoid this problem by compiling a voluminous undifferentiated collection of ad-hoc operations from the literature.

Three basic operations, however, are **or**, **and**: $[0,1]^2 \to [0,1]$ and **not**: $[0,1] \to [0,1]$. If one accepts the axioms that **or** and **and** should be monotone, continuous, commutative and associative operations which distribute over each other and satisfy $(a \textbf{ and } b) \leqslant a \leqslant (a \textbf{ or } b)$, $(0 \textbf{ or } 0) = 0$, $(1 \textbf{ and } 1) = 1$ then there is a unique such pair of operations, namely $a \textbf{ or } b = \max(a, b)$, $a \textbf{ and } b = \min(a, b)$. (It should be pointed out, however, that these operations were introduced in a very similar spirit in 1930 by Łukasiewicz and Tarski [**15**]. Excepting vague references (such as those on p. 151) the authors give scant evidence of having researched pre-1960 literature.) Much more problematic is $\textbf{not}(a) = 1 - a$, a definition subject to some controversy in the literature. In the framework of §4 below, I shall argue that this definition is more natural for probability theory than for fuzzy set theory (in a suitably precise sense).

The *extension principle* asserts that each function $f\colon X_1 \times \cdots \times X_n \to [0,1]^Y$ extends to $\tilde{f}\colon [0,1]^{X_1} \times \cdots \times [0,1]^{X_n} \to [0,1]^Y$ by

$$\text{(A)} \quad \tilde{f}(\mu_1,\ldots,\mu_n)(y) = \sup_{x_1,\ldots,x_n} \min(\mu_1(x_1),\ldots,\mu_n(x_n), f(x_1,\ldots,x_n)(y)).$$

This reduces to the special case

$$\text{(B)} \qquad \tilde{f}(\mu_1,\ldots,\mu_n)(y) = \sup_{f(x_1,\ldots,x_n)=y} \min(\mu_1(x_1),\ldots,\mu_n(x_n)),$$

when $f\colon X_1 \times \cdots \times X_n \to Y$ (i.e., embed Y in $[0,1]^Y$ by mapping y to the characteristic function of $\{y\}$). Mathematicians do not usually feel that the existence of such formulas deserves to be called a 'principle'; better justification is attempted in §4 below.

Using the extension principle, a group structure $m\colon X \times X \to X$ and a metric structure $d\colon X \times X \to \mathbf{R}^+$ 'extend' to $\tilde{m}\colon [0,1]^X \times [0,1]^X \to [0,1]^X$ and $\tilde{d}\colon [0,1]^X \times [0,1]^X \to [0,1]^{\mathbf{R}^+}$. Paraphrasing from p. 38, variables which have values have been generalized to variables which assign a degree of membership in $[0,1]$ to each value. But there is a clash here with the original motivations. We were originally led to believe that 'classical is to fuzzy as $\{0,1\}$ is to $[0,1]$'. But we would expect the 'classical extension' of m and d to be themselves whereas, however, (B) restricts to the familiar extension of functions to subsets

$$\text{(B}'\text{)} \qquad \tilde{f}(A_1,\ldots,A_n) = f(A_1 \times \cdots \times A_n)$$

(identifying $\{0,1\}^X$ with the set of subsets of X). Thus the classical extension of a metric space associates to a pair of subsets A, B the set $\{d(a,b)\colon a \in A, b \in B\}$. In my opinion this deglamorizes the discussion on p. 39 which asserts that the fuzzy extension of d 'models a distance between fuzzy spots'. What is missing is a suitable function $[0,1]^{\mathbf{R}^+} \to \mathbf{R}^+$ such as infimum which, in the classical case, would yield a more suitable notion of distance between sets. I am surprised that the need for such functions has not been recognized in fuzzy set theory. A candidate is introduced below in §4.

A number of other concepts involve sufficient additional structure so as to render it artificial to classify them as within fuzzy *set theory*. A *fuzzy number in* $\mathbf{R}^n$ is a fuzzy set $\mu\colon \mathbf{R}^n \to [0,1]$ which is piecewise continuous, achieves the value 1 at least once, and is such that $\{x\colon \mu(x) \geqslant \alpha\}$ is convex for each α. The algebraic structure of the set of fuzzy numbers in **R** is a personal favorite of the authors and makes several appearances. Some readers might wish to investigate the *Sugeno integral* of an interval-valued function over a subset.

2. Theories that exist. The theory of probability and statistics is both a rich area of pure mathematics and a cornerstone of experimental science. The ease with which this theory is applicable to real world problems has created an atmosphere which suggests that all phenomena must be either deterministic or probabilistic. As an illustration of this attitude, I quote from the historic-philosophic prelude of a text on the foundation of quantum mechanics [**9**, p. 71], calling attention to the third sentence.

> "...If a system S is subject to conditions $A, B, \ldots$ then the effects $X, Y, \ldots$ can be observed. In this form it establishes a relation between the conditions and the effects.
>
> The most general relation of this kind which can be formulated is a *probability relation*."

Some of the relationships between fuzzy set theory and probability theory are explored on pp. 136–146. Goodman [**7**] has asserted that fuzzy set theory is subsumed by probabilistic concepts.

In the preface to *Theories of probability*, [**5**], Fine states (the parenthetical remark is my own)

> "Efforts to understand, usefully formulate, and resolve the problems encountered (by electrical engineers) in the design and analysis of inference and decision-making systems led, it now seems inexorably, to a study of the foundations of probability".

In this book, Fine considers a number of alternative theories of nondeterminism which, in my judgment, are in fact not probability theories. Three of these are

(a) Comparative probability, an axiomatization of the relation "A is at least as probable as B".

(b) A concept of 'randomness' based on the notion of pseudorandom sequences with high computational complexity. A 'pseudorandom sequence' results from a deterministic algorithm which from the classical point of view 'simulates a random sequence' (see [**11**] for a mathematical discussion). But the intent here is to bypass dependence on the classical theory. This idea is due to Kolmogoroff [**13**]. (As a tangential observation, the concept of 'computational complexity' used, namely the length of a description of the generating algorithm, is much at odds with more recent work in complexity [**16**].)

(c) Logical probability. This is a form of modal logic arising by adjoining to classical predicate logic the modal operator "it is probably true that...". See [**3**].

At the time of Fine's book, any of these theories enjoyed a state of mathematical development at least commensurate with the current state of fuzzy set theory. Points of contact with these theories were not explored in the book under review. Perhaps calling something a 'probability theory' automatically purges it from the fuzzy camp. As Bellman has said [**2**, p. 32]

> "We all wear such intellectual blinders it is amazing that anything new is every developed".

Why is fuzzy set theory an appealing idea? Well, we mathematicians for the most part have been taught that mathematical structures are built on sets and that the language of set theory underlies the rigorous foundation of all the work that we do. Fuzzy set theory hopes to deal with 'imprecision' and 'vagueness' (e.g. as they arise in the experimental sciences) by inventing new sets which are intrinsically imprecise or vague, at the same time allowing the familiar external operations used to build new sets from old. A fuzzy philosophers' stone is sought that will routinely 'fuzzify' existing mathematics.

Probability theory is not tailored to such a program. *Topos theory*, on the other hand, is a mathematical area in a high state of development [**6, 8, 10**] which springs from the very similar touchstone of providing 'intrinsically variable sets' [**14**]. The contrast in style between topos theory and fuzzy set theory is profound. Whereas fuzzy set theory proceeds by ad-hoc imitation of standard set theory (which may be like trying to discover finite fields by imitating $\mathbf{Z}_2$), topos theory creates refreshing structural analogies at a deeper level. The only real axiom on a topos posits a precise sense in which a subset R of $X \times Y$ may be recast as the subset-valued function $x \mapsto \{y: xRy\}$. Virtually all of the standard set-theoretic constructions can be deduced from this axiom. The methods used provide an inspiration for all workers hoping to build new set theories. For further contrast between topos theory and fuzzy set theory see [**19**].

3. Four criteria for a new set theory. (i) *Describe a class of theories.* More precisely, rather than relying upon a single ad-hoc imitation of classical set theory, single out as a paradigm particular properties considered important and investigate the class of all set theories that enjoy such structure. This includes the development of tools to compare theories.

(ii) *Mainstream examples.* Usual models of set theory should be included in an unambiguous way. The motivating paradigm or other well-known structures may suggest other examples.

(iii) *Internal development.* The motivating paradigm should have conceptual richness, revealed by showing that each model carries an interpretation of important notions without additional axiomatization. Possible such notions are the equality of sets and the logical structure of the set of subsets.

(iv) *External development*. This includes classifications through the use of additional axioms, representation theorems (e.g. relating an abstract class to the mainstream examples) and characterization theorems.

Topos theory meets these criteria admirably. Since the papers of Cole and Mitchell [**4, 20**] the class of toposes coextensive with earlier first-order models of set theory has been well understood and, in any case, that everyday set theory is a topos was clear from the beginning. A special strength of topos theory is that the category of sheaves on a topological space is also a mainstream example, so that topos theory has offered a rapprochement between set theory and sheaf theory. Each viewpoint induces its own notion of homomorphism between toposes, leading to a rich theory of comparison. The internal development of topos theory includes reasonable facsimiles of all constructions of higher-order set theory and its underlying logic (which is intuitionistic in general). The wealth of external development is hinted at by quoting two papers, [**14, 12**] in that order.

> "Around 1963...five distinct developments in geometry and logic became known, the subsequent unification of which has, I believe, forced upon us the serious consideration of a new concept of set. These were the following:
> 'Non-Standard Analyis' (A. Robinson)
> 'Independence proofs in Set Theory' (P. J. Cohen)
> 'Semantics for Intuitionistic Predicate Calculus' (S. Kripke)
> 'Elementary Axioms for the Category of Abstract Sets' (F. W. Lawvere)
> 'The General Theory of Topoi' (J. Giraud)"

> "We attempt here to present a foundation of a kind of Differential Algebra, where the differentiation process is not an added *structure*, but something which stems from a *property* of the ring object considered. Ring objects of this kind ('rings of line type') are not present in the category of sets, but occur in some of the toposes of algebraic geometry, as well as in the category of formal schemes."

There is a fifth criterion that a fuzzy set theory is expected to have which not every set theory would: wide-spectrum applicability to modelling of nondeterminism in engineering and experimental science. This rules out topos theory which, at least to date, has primarily addressed issues of mathematical foundation.

4. Distributional set theories. In this section we introduce a class of set theories which address the four criteria of the previous section and also provide a backdrop for delineating fuzzy set theory (which is a particular case). Mathematical details and historical remarks appear in [**18**], although the motivating paradigm is different there.

We begin with the viewpoint that a fuzzy set μ with universe X is a 'distribution' on X. The special case in which a distribution is a subset (μ takes values 0 or 1 only) we shall call *possibilistic set theory*. The further special case of *crisp set theory* restricts to singleton subsets.

Motivating paradigm. Given sets n, X, a distribution ω on n and a family $(\mu_i$: $i \in n)$ of distributions on X induces a 'net distribution' on X. In the case of fuzzy set theory it is clear what we should try: $\omega(i)$ is the 'degree of belief μ_i is chosen' and $\mu_i(x)$ is the 'degree of belief x occurs given μ_i' so that, since **or** is sup and **and** is inf in fuzzy set theory, the 'net degree of belief x occurs' is $\sup_i \min(\omega(i), \mu_i(x))$. Let us glorify this with a name and a notation:

ω-unions for fuzzy sets. Each function ω: $n \to [0,1]$ induces, for each set X, the n-ary operation $([0,1]^X)^n \to [0,1]^X$ of *ω-union*, denoted $(\mu_i$: $i \in n) \mapsto \bigsqcup_{\omega \in i} \mu_i$ and defined by

$$\Big(\bigsqcup_{\omega \in i} \mu_i\Big)(x) = \sup_i \min(\omega(i), \mu_i(x))$$

The following result, not noted in the literature of fuzzy set theory to my knowledge, makes the 'extension principle' a principle:

Extension principle. Embed X in $[0,1]^X$ via characteristic functions of singletons. Then the function $\tilde{f}$ of (A) is the unique extension of f which preserves arbitrary ω-unions in each variable separately.

Thus assured of being on the right track, the formal definition is as follows.

Distributional set theories. A distributional set theory is $\mathbf{T} = (T, e, (-)^{\#})$ where T assigns to each set X a set TX (of 'distributions on X'), e assigns to each set X a function e_X: $X \to TX$ ('point distributions') and $(-)^{\#}$ assigns to each n-tuple α: $n \to TX$ of distributions on X a function $\alpha^{\#}$: $Tn \to TX$ (so that $\alpha^{\#}(\omega)$ is the 'net distribution' of the motivating paradigm) subject to the following three axioms:

(i) $\alpha^{\#} e_X = \alpha$.

(ii) $(e_X)^{\#} = \mathrm{id}_{TX}$.

(iii) If β: $X \to TY$, $(\beta^{\#}\alpha)^{\#} = \beta^{\#}\alpha^{\#}$.

The first axiom asserts that if $\omega \in n$ is a point distribution, the net distribution induced by $(\mu_i$: $i \in n)$ is μ_ω. The second axiom guarantees that if μ_i is the point distribution on i ($i \in n$), the net distribution induced by ω is ω itself. Despite its initial technical appearance, the third axiom states, simply, that composition of **T**-relations is associative. Here, a **T**-*relation from X to Y* is a function β: $X \to TY$. If α is a **T**-relation from n to X it should 'compose' with β to produce a **T**-relation $\beta \circ \alpha$ from n to Z. Define $\beta \circ \alpha = \beta^{\#}\alpha$. Axiom (iii) is equivalent to $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$. (The $\beta \circ \alpha$ construction for fuzzy sets appears on p. 99.)

Crisp set theory is an example with $TX = X$, $e_X = \mathrm{id}_X$, $\alpha^{\#} = \alpha$. Possibilistic set theory has $TX = 2^X$, $e_X(x) = x$, $\alpha^{\#}(A) = \bigcup(\alpha(a)$: $\alpha \in A)$. For fuzzy set theory, $TX = [0,1]^X$, $e_X(x) = \chi_{\{x\}}$, and $\alpha^{\#}(\omega)$ is the ω-union of α.

Comparison of theories is via *theory maps* λ: $\mathbf{T} \to \bar{\mathbf{T}}$ which are families of functions λ_X: $TX \to \bar{T}X$ subject to two axioms [**18**, Definition 1.10]. Such λ is a *subtheory of* $\overline{\mathbf{T}}$ if each λ_X is injective. Possibilistic set theory is a subtheory of fuzzy set theory. With the exception of two 'inconsistent theories', crisp set theory is a subtheory via e_X: $X \to TX$. The notion of a 'cut point' $0 \leqslant c < 1$, often used in papers on fuzzy set theory to project [0, 1] onto $\{0, 1\}$, is aptly described as a theory map $[0, 1]^X \to 2^X$ from fuzzy set theory to possibilistic set theory.

Examples of distributional set theories abound. One mainstream example is *probabilistic set theory* for which TX is the set of finite-support probability distributions on X, $e_X(x)$ is the usual point distribution and $\alpha^{\#}(\omega)$ assigns probability $\Sigma_i\, \omega(i)\mu_i(x)$ to each $x \in X$. Although 'finite nonempty subsets' is a subtheory of possibilistic set theory, this theory cannot be identified with 'equally likely probability' since mapping an n-element subset to the probability distribution assigning each member $1/n$ is not a theory map. Another example is 'm-flou sets' (pp. 28, 29) which is $TX = \{(E_1, \ldots, E_m)$: $E_1 \subset \cdots \subset E_m \subset X\}$, $e_X(x) = (\phi, \ldots, \phi, x)$, $\alpha^{\#}(E_1, \ldots, E_m) = (F_1, \ldots, F_m)$ where, if $\alpha(x) = (F_{1x}, \ldots, F_{mx})$, $F_i = \bigcup(F_{ix}\colon x \in E_i)$. The underlying concept of LeFaivre's programming language FUZZY (p. 266) is partly captured by the following distributional set theory. Fix a 'credibility partially-ordered set' $(P, \leqslant)$ with binary infima and greatest element 1. Define $TX = X \times P$('$(x, p) = x$ with credibility p'), $e_X(x) = (x, 1)$, and, given α: $n \to TX$ so that $\alpha(x) = (f(x), s(x))$, $\alpha^{\#}(x, p) = (f(x), \inf(p, s(x)))$.

In the standard approach to fuzzy set theory, [0, 1] is chosen as a set of 'truth values' generalizing $2 = \{0, 1\}$, fuzzy sets are defined explicitly as functions $X \to [0, 1]$ and the logical operations **or, and** and **not** on [0, 1] then extend pointwise to fuzzy sets. We now contrast this situation with some aspects of the internal development of a distributional set theory **T**. (It will take a few paragraphs to explain how to treat the Boolean operations.)

The crisp truth set is 2; define the set of **T**-*truth values* to be $T2$. In crisp set theory, a 'point' is a one-element set (which we denote hence as 1). A **T**-*point* is an element of $T1$. There is the immediate discrepancy that for **T** = fuzzy set theory, [0, 1] is identified with the set of points rather than with the set of truth values; the latter is $[0, 1] \times [0, 1]$ whose elements may be interpreted as 'independent truth and falseness values'. For **T** = probabilistic set theory, $T2$ is identified with [0, 1] (via 'probability of true').

For general **T**, for $x \in X$ and $\mu \in TX$, define the *degree of membership of* x *in* μ as $dm(x, \mu) = \alpha_x^{\#}(\mu) \in T2$, where $\alpha_x = e_2\chi_{\{x\}}$. Each distribution $\mu \in TX$ is then *represented* by $dm(-, \mu)$: $X \to T2$. **T** is *faithful* if $\mu \mapsto dm(-, \mu)$ is injective, as is the case for all theories so far mentioned, but for none of these is $TX \to T2^X$ surjective. Call elements of $T2^X$ **T**-*propositions on* X. Thus explicit representation of distributions as propositions follows from more basic axioms, but propositions are more general than distributions. In fuzzy set theory, $dm(-, \mu)(x) = (\mu(x), \bar{\mu}(x))$ where $\bar{\mu}(x) = \sup_{x \neq y} \mu(y)$. In probabilistic set theory, $dm(-, \mu)$ is μ itself. Curiously, probabilistic proposition = fuzzy set.

The generalization of ω-unions is immediately at hand, since each $\omega \in Tn$ induces the operation $(TX)^n \to TX$, $(\mu_i\colon i \in n) \mapsto \alpha^{\#}(\omega)$ where $\alpha(i) = \mu_i$. Call such operations **T**-*operations*. The **T**-*extension principle* that each f: $X_1 \times \cdots \times X_n \to TY$ has a unique extension $\tilde{f}$: $TX_1 \times \cdots \times TX_n \to TY$ which preserves all **T**-operations in each variable separately characterizes the *commutative* theories which are so called by virtue of being equivalently characterized by the property that any two **T**-operations commute with each other. All examples mentioned so far are commutative. For commutative theories, the image of the extension $TX \times TY \to T(X \times Y)$ of $e_{X \times Y}$ constitutes the 'independent joint distributions'. This suggests that a distributional set theory appropriate for a 'quantum set theory' would not be commutative.

In contrast to the ad-hoc approach of pp. 160–173, the 'logic of propositions' is subject to internal development, at least for commutative theories. The Boolean operations on 2 extend to $T2$ and hence, by pointwise operations, to each proposition space $T2^X$. In general such operations need not map distributions to distributions which is hardly surprising since, in crisp set theory, Boolean operations do not map singletons to a singleton. For fuzzy set theory, the extension of **not**: $2 \to 2$ to $[0, 1] \times [0, 1]$ is $(a, b) \mapsto (b, a)$ whereas in probabilistic set theory the extension is $[0, 1] \to [0, 1]$, $a \mapsto 1 - a$. This explains the earlier claim that $1 - a$ is more naturally associated with probability theory.

A further aspect of internal development concerns 'equality of distributions'. Given $\mu, \mu' \in TX$, define $\text{eq}(\mu, \mu') = dm(-, \mu)^{\#}(\mu') \in T2$. (For commutative theories, $\text{eq}(\mu, \mu') = \text{eq}(\mu', \mu)$.) For possibilistic set theory, interpret the four truth values ϕ, $\{0\}$, $\{1\}$, 2 respectively as 'undefined', 'no', 'yes', and 'maybe' whence $\text{eq}(A, A')$ is undefined if either set is empty and otherwise is no if the sets are disjoint, is yes if both equal the same singleton and is otherwise maybe. The *consistency* of two fuzzy sets μ, μ', with universe X has been defined by Zadeh (p. 24) as $C(\mu, \mu') = \sup_x \min(\mu(x), \mu'(x))$, and (p. 25) the *separation index* is defined by $1 - C(\mu, \mu')$. For fuzzy set theory, $\text{eq}(\mu, \mu')$ has true coordinate $C(\mu, \mu')$ but has false coordinate $\sup_{x \neq y} \min(\mu(x), \mu'(y))$ which would then be seen as the candidate for the separation index from the distributional point of view.

The external development of distributional set theories includes a number of simple notions for classification such as *crisp points* (e_1: $1 \to T1$ is an isomorphism), *noise-free* ($T\phi = \phi$), *antireflexive* ($\text{eq}(\mu, \mu) =$ true implies μ is crisp) and the *eigenstate condition* (if $d\mu(x, \mu) =$ true then $\mu = e_X(x)$). It is a theorem that crisp points and consistent implies noise-free. Any theory has a largest subtheory with crisp points. For commutative theories, having crisp points has two equivalent characterizations, namely (a) the independent joint distributions map $TX \times TY \to T(X \times Y)$ is injective and (b) the equation x **and** $0 = 0$ lifts from 2 to $T2$.

Still promised from §1 is the concept of a map $TX \to X$ which 'averages over' a distribution in a manner consistent with the algebraic structure of **T**. A **T**-*decider* is (X, ξ) where ξ: $TX \to X$ satisfies the axioms

(i) $\xi e_X = \text{id}_X$.

(ii) Given α, β: $n \to TX$ with $\xi\alpha = \xi\beta$, $\xi\alpha^{\#} = \xi\beta^{\#}$.

The first axiom is eminently reasonable, assuring that the only value allowed to represent the point distribution on x is x itself. The second axiom states that ξ of the net distribution induced by $\omega \in Tn$ and $\mu_i \in TX$ $(i \in n)$ depends only on ω and the $\xi(\mu_i)$. A *decider homomorphism* $f: (X, \xi) \to (Y, \theta)$ is a function $f: X \to Y$ such that $\theta(Tf) = f\xi$, where $Tf: TX \to TY = (e_Y f)^\#$. (The Tf construction generalizes the 'fuzzy extension of a nonfuzzy function' of p. 98.) For any set X, $(TX, \mathrm{id}^\#_{TX})$ is a **T**-decider, indeed is the *free decider generated by* X in that for any **T**-decider (Y, θ) and any function $g: X \to Y$, there exists a unique decider homomorphism $g^\#: (TX, \mathrm{id}^\#_{TX}) \to (Y, \theta)$ with $g^\# e_X = g$, namely $g^\# = \theta(Tg)$. The older $\alpha^\#: TX \to TY$ is just such an extension, so the notations do not clash.

One class of theories for which the deciders are understood is the class of *matrix theories* over a complete partial semiring [**18**, Definition 7.4]. Rather than elaborate, two examples should suffice. For fuzzy set theory, the complete semiring is $[0, 1]$ with (infinite) sum = sup and (binary) multiplication = inf. Here $\alpha: X \to [0, 1]^Y$ may be viewed as a matrix with entries in the semiring, X indexing columns and Y indexing rows. The composition $\beta \circ \alpha = \beta^\# \alpha$ is then matrix multiplication and e_X acts as the identity matrix. Similarly, possibilistic set theory is the matrix theory over the two-element complete subsemiring of $[0, 1]$, $\{0, 1\}$. For matrix theories, decider = module. In particular, TX = free module. This leads one to view an element μ of $[0, 1]^X$ as the formal linear combination

$$\mu = \sum_{x \in X} \mu(x) x.$$

This is more transparent in the restriction to possibilistic set theory wherein a subset is viewed as the union of its singletons. For possibilistic set theory, decider = complete semilattice in that if $(X, \leqslant)$ is a complete semilattice then sup: $2^X \to X$ is a decider structure, whereas, conversely, if $\xi: TX \to X$ is a decider then $x \leqslant y$ defined by $\xi\{x, y\} = y$ defines a complete semilattice whose supremum operator is ξ. In particular, the infimum map $2^{\mathbf{R}^+} \to \mathbf{R}^+$ needed in §1 is just the decider structure of $(\mathbf{R}^+, \geqslant)$.

5. By way of conclusion. I hope the reader has not gained the impression that all work in fuzzy set theory is without purpose, since much of the work reported in the book under review is of obvious use in the engineering sciences, regardless of whether or not any 'fuzzy set theory' is involved. The focus of this review is on a concern that the mathematical theory advocated by fuzzy set theorists is misguided. I do not seriously feel that any one approach such as topos theory or distributional set theory is the only way to advance a theory of inherently imprecise sets. There is every reason to expect a new generation of fuzzy set theorists. For them my advice is: study what other mathematicians have done and then build a beautiful theory!

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Locally convex spaces, by H. Jarchow, Teubner, Stuttgart, 1981, 14 + 534 pp., DM 98,--.

Modern methods in topological vector spaces, by Albert Wilansky, McGraw-Hill, 1979, xiii + 298 pp., $39.95.

In the preface to the 1973 English translation [4] of his 1954 notes on locally convex spaces, Grothendieck wrote to the effect that the translation was verbatim and that no attempt had been made to update the notes since *nothing had happened in the theory of locally convex spaces* since the appearance of his notes twenty years earlier.

In his 1976 review of the translation, John Horváth wrote [8] "Even after twenty years, Grothendieck's book is an elegant and refreshing introduction to topological vector spaces, and in spite of the fact that at least *ten* monographs have been written on the subject since 1954, it is probably the best text book to use in a course."

Horváth's own book on the subject [9] appeared in 1966!

Continuing with Horváth on Grothendieck: "The proofs are at times concise or even omitted, but this enhances its value as a text book. An additional