

EUCLIDEAN, HYPERBOLIC AND SPHERICAL BLOCH CONSTANTS

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TO PROFESSOR S. E. WARSCHAWSKI ON HIS 78TH BIRTHDAY

We begin with a brief survey of some of the known results dealing with Bloch constants. Bloch's theorem [3] asserts that there is a constant $B_1 > 0$ such that if f is holomorphic in the open unit disk \mathbf{B} and normalized by $|f'(0)| \geq 1$, then the Riemann surface of f , viewed as spread over the complex plane \mathbf{C} , contains an unramified disk of radius at least B_1 . Pommerenke [11] introduced the locally schlicht Bloch constant $B_\infty > B_1$ which has the same property relative to the family of normalized locally schlicht holomorphic functions defined on \mathbf{B} . He showed that $B_\infty \leq \wp$, where \wp denotes the Landau constant [7]. The precise values of these constants are not known; however, the following bounds are known.

$$.433 < \frac{\sqrt{3}}{4} < B_1 \leq \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)} < .4719,$$
$$\frac{1}{2} < B_\infty \leq \wp \leq \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)} < .5433.$$

The lower bounds for B_1 and B_∞ , but without strict inequality, are the work of Ahlfors [1]. The strict inequalities were established by Heins [6] and Pommerenke [11], respectively. The upper bound for the Bloch constant comes from an example of Ahlfors and Grunsky [2] and is conjectured to be sharp. The upper bound for the Landau constant that is frequently cited is .544; this is due to an unpublished example of R. M. Robinson that is mentioned in [1]. In [8] we present an explicit example which yields the upper bound for the Landau constant that is given above. It is analogous to the Ahlfors-Grunsky example and it is plausible that it should give the actual value of the Landau constant. It is probably the example of Robinson. In fact, in [8] we exhibit a unified approach to obtaining upper and lower bounds for these and other new Bloch constants. As special cases of our results we obtain all of the previously mentioned bounds.

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Before we can define the various Bloch constants, we must establish certain notation and terminology. Let \mathbf{P} denote the Riemann sphere. Suppose $f: \mathbf{B} \rightarrow \Omega$ is a meromorphic function, where Ω is one of \mathbf{B} , \mathbf{C} and \mathbf{P} . Of course, if Ω is \mathbf{B} or \mathbf{C} , then f is actually holomorphic. We view R_f , the Riemann surface of f , as being spread over Ω . In order to measure the size of unramified disks on R_f , we need to fix a distance function on Ω . Set

$$\begin{aligned} \delta_{\mathbf{B}}(z, w) &= \left| \frac{z - w}{1 - \overline{w}z} \right| && \text{if } z, w \in \mathbf{B}, \\ \delta_{\mathbf{C}}(z, w) &= |z - w| && \text{if } z, w \in \mathbf{C}, \\ \delta_{\mathbf{P}}(z, w) &= \left| \frac{z - w}{1 + \overline{w}z} \right| && \text{if } z, w \in \mathbf{P}. \end{aligned}$$

Then $\delta_{\mathbf{B}}$ and $\delta_{\mathbf{C}}$ are true distance functions while $\delta_{\mathbf{P}}$ is not. However, $2 \tan^{-1}(\delta_{\mathbf{P}}(z, w))$ does give the spherical distance between z and w , so $\delta_{\mathbf{P}}$ is closely related to a distance function. In any case, we shall use δ_{Ω} ($\Omega = \mathbf{B}$, \mathbf{C} or \mathbf{P}) to measure the radius of disks on Ω . The reason for doing this is that the close analogy between bounds for the various Bloch constants will be apparent. For $z \in \mathbf{B}$ let $r_{\Omega}(z, f)$ denote the radius, relative to δ_{Ω} , of the largest unramified disk in R_f having center $f(z)$. Define $r_{\Omega}(f) = \sup\{r_{\Omega}(z, f): z \in \mathbf{B}\}$. Set

$$f^{\Omega}(0) = \begin{cases} \frac{|f'(0)|}{1 - |f(0)|^2} & \text{if } \Omega = \mathbf{B}, \\ |f'(0)| & \text{if } \Omega = \mathbf{C}, \\ \frac{|f'(0)|}{1 + |f(0)|^2} & \text{if } \Omega = \mathbf{P}. \end{cases}$$

For $m \in \mathbf{Z}^+ \cup \{\infty\}$ let $F_m(\mathbf{B}, \Omega)$ denote the family of all nonconstant meromorphic functions $f: \mathbf{B} \rightarrow \Omega$ such that for each $q \in f(\mathbf{B})$ each root of $f = q$ is either simple or else has multiplicity at least $m + 1$. For $m = 1$ this is just the family of all nonconstant meromorphic mappings of \mathbf{B} into Ω , while for $m = \infty$ this gives locally schlicht meromorphic functions.

Now we can introduce the various Bloch constants. For $\alpha > 0$ let

$$\mathcal{B}_{m, \Omega}(\alpha) = \inf\{r_{\Omega}(f): f \in F_m(\mathbf{B}, \Omega) \text{ and } f^{\Omega}(0) \geq \alpha\}.$$

It is straightforward to check that $\mathcal{B}_{m, \mathbf{C}}(\alpha) = \alpha \mathcal{B}_{m, \mathbf{C}}(1)$. Notice that $\mathcal{B}_{1, \mathbf{C}}(1)$ is the classical Bloch constant and that $\mathcal{B}_{\infty, \mathbf{C}}(1)$ is the locally schlicht Bloch constant. In the hyperbolic case ($\Omega = \mathbf{B}$) or the spherical case ($\Omega = \mathbf{P}$), the Bloch constants are not linear functions of α .

We obtain lower bounds for these hyperbolic, euclidean and spherical Bloch constants. These bounds are established by making use of a differential-geometric method that was introduced by Ahlfors [1] and refined by Heins [6]. The main ingredient is a generalization of Schwarz' Lemma to metrics. The method was originally employed in the context of euclidean Bloch constants. We adapt the method so that it works in hyperbolic or spherical geometry. The essential features of the adaptation to spherical geometry occur in the work of Pommerenke [10]. His approach is analytic, while ours is geometric. In [8] we show that $\mathcal{B}_{m,\Omega}(\alpha) > h_{m,\Omega}^{-1}(\alpha)$, where $h_{m,\Omega}^{-1}(\alpha)$ is an explicit function of α . We give selected values of these functions below.

$$h_{m,\mathbf{B}}^{-1}(\alpha) = \begin{cases} \alpha \left[\frac{3}{8 - 5\alpha^2 + 4(\alpha^4 - 5\alpha^2 + 4)^{1/2}} \right]^{1/2}, & m = 1, \\ \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}, & m = \infty, \end{cases}$$

$$h_{m,\mathbf{C}}^{-1}(\alpha) = \begin{cases} \frac{\alpha\sqrt{3}}{4}, & m = 1, \\ \frac{\alpha}{2}, & m = \infty, \end{cases}$$

$$h_{m,\mathbf{P}}^{-1}(\alpha) = \begin{cases} \alpha \left[\frac{3}{8 + 5\alpha^2 + 4(\alpha^4 + 5\alpha^2 + 4)^{1/2}} \right]^{1/2}, & m = 1, \\ \frac{\alpha}{1 + \sqrt{1 + \alpha^2}}, & m = \infty. \end{cases}$$

Actually, in [8] these lower bounds are established in the context of meromorphic functions $f: X \rightarrow \Omega$, where X is any hyperbolic Riemann surface and f is suitably normalized at a fixed point of X . Also, the lower bounds for spherical Bloch constants improve bounds due to Tsuji [12], Greene and Wu [4] and Jorgenson [5].

We obtain upper bounds for the various Bloch constants by modifying the example of Ahlfors and Grunsky [2] to suit the geometry of the situation at hand. We refer the reader to [8] for details. We just note that for the hyperbolic and spherical Bloch constants examples are now needed for every admissible value of α rather than just for $\alpha = 1$ as is the case for euclidean Bloch constants.

These results have applications to meromorphic functions defined on \mathbf{C} . Let

$$\mathcal{B}_{m,\Omega} = \inf \{r_\Omega(f) : f \in F_m(\mathbf{C}, \Omega)\}.$$

Of course, here we are only considering $\Omega = \mathbf{C}$ or \mathbf{P} . It is known that $\mathcal{B}_{1,\mathbf{C}} = \infty$; that is, the Riemann surface of a nonconstant entire function contains arbitrarily large unramified disks. In [9] we show that

$$\frac{1}{\sqrt{3}} \leq \mathcal{B}_{1,\mathbf{P}} \leq \frac{1}{\sqrt{2}}, \quad \mathcal{B}_{\infty,\mathbf{P}} = 1.$$

We also obtain results for Bloch constants for meromorphic functions on compact Riemann surfaces in [9].

ADDED IN PROOF. Recently, the author was informed that E. Peschl (*Über unverzweigte konforme Abbildungen*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **185** (1976), 55–78) had also considered euclidean, hyperbolic and spherical Bloch constants for locally schlicht functions in the unit disk and that he had obtained the nonstrict inequality $\mathcal{B}_{\infty,\Omega}(\alpha) \geq h_{\infty,\Omega}^{-1}(\alpha)$ by a different method.

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