A major issue ($2\frac{1}{2}$ chapters worth) is the stability of factors $T_1$ and $T_2$ with respect to perturbations of $T$. Conversion to an invariant subspace problem yields satisfactory results on the issue of which $T$ have stable factorizations.

An intriguing observation is an explicit correspondence between factorizations of $T$ and the solutions of an algebraic Ricatti equation. They use this to study stable solutions to the Ricatti equations. Related relationships I have seen (in the engineering literature) are in comparatively special circumstances.

Most of the material in the book fits easily into infinite dimensional space and that is where it is done. The authors are consequently able to study certain integro-differential equations. In particular their methods apply to the transport equation (of nuclear physics) and a chapter is devoted to this. This approach to the transport equation has proved to be valuable and the interested reader should see a forthcoming book on transport equations to appear in the same Birkhäuser series.

There are many other nice ideas which cannot be mentioned in a brief review. In summary, the first third of the book sets out principles of model and system theory of such general interest that it could serve as an introduction to many readers. It does not give physical motivation or many references to the systems literature, so the beginner would want a more engineering oriented supplement (e.g. T. Kailath's book). Also to fill in more model theory, one could see either the definitive book of Nagy and Foiaş or the more informal account of the Nagy-Foiaş theory, by R. G. Douglas, which is contained in the volume of the MAA studies series which C. Pearcy edited. Also there is Brodskii's book. The remainder of the book is also accessible with little background and contains much fine mathematics.

J. WILLIAM HELTON

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 2, March 1982

0273-0979/81/0000-0292/$03.25

*Spline functions*: *Basic theory*, by Larry L. Schumaker, Wiley, New York, 1981, xiv + 553 pp., $42.50.

A polynomial spline function results from splicing polynomial arcs in such a way that the resulting function is sufficiently smooth. In more precise language, a polynomial spline function of degree $k \geqslant 0$ is a real function defined by piecewise polynomial components of degree $\leqslant k$ whose derivatives through order $k - 1$ are continuous. The juncture points are commonly referred to as (simple) *knots* in the literature. Central to the study of these functions is the class of *minimal support splines* or *B-splines*. It is found that the smallest possible number of knots of a spline of degree $k$ whose support is a compact interval in the interior of its domain is $k + 2$. Such splines are uniquely determined up to constant multiples. They are ideal basis functions and can be calculated recursively by formulas which express a $B$-spline of a given degree $k$ as a convex linear combination of two $B$-splines of degree $k - 1$.

Differentiation and integration satisfy similar recursions. The $B$-splines also have revealing geometric and probabilistic interpretations and may constitute the fundamental device for formulation of an adequate multi-dimensional theory of splines going beyond simple tensor products on rectangular partitions or piecewise polynomials on simplicial decompositions. This has not taken place, however, beyond some important first steps.

Applications to interpolation, quadrature, function approximation, and the numerical solution of differential equations very rapidly led to the extension of the notion of a spline from that given above. Thus, by the latter 1960's, when the reviewer began to work in this field, such terminology as generalized spline and deficient spline with multiple knots had already entered the literature, a reflection of the desire to relax derivative continuity requirements at the knots in the latter case, and to substitute for polynomial components, in the former case, general finite dimensional systems, such as Tchebycheff systems. By this time the subject was already in its third decade, having been formally initiated in a classic paper by I. Schoenberg **[1946]**.[1] The subject had developed surprisingly slowly in its early stages, and was largely ignored, except implicitly, in the surge of books dealing with approximation theory and appearing during the period 1962–1967 (cf. M. Golomb **[1962]**, P. Davis **[1963]**, A. Sard **[1963]**, A. Timan **[1963]**, J. Rice **[1964]**, E. Cheney **[1966]**, G. Lorentz **[1966]**, P. Butzer and H. Berens **[1967]** and G. Meinardus **[1967]**). It was not until a stratum of optimality and stability results had accumulated, however, that unifying veins were perceived and tapped. Such a systematic development was occurring when the author of the book under review began his tenure at the Mathematics Research Center of the University of Wisconsin in 1966, having finished a dissertation under S. Karlin at Stanford, in which he satisfactorily resolved an open existence question related to a 1958 conjecture of Schoenberg on the validity of the fundamental theorem of algebra for monosplines; these appear, for example, as kernels in quadrature remainder formulas and are individually represented as a monomial of a given degree perturbed by a spline of degree one less. The author's result involved the delicate structure of total positivity, begun by G. Polya and carried on by I. Schoenberg, S. Karlin and others, and of Haar or Tchebycheff systems (cf. S. Karlin and W. Studden **[1966]** and S. Karlin **[1968]**). The reader may conveniently think of Tchebycheff systems of order $n$ as characterized by the unique interpolation property of $n$ arbitrary data at $n$ arbitrary points. The spline systems themselves constitute what are termed weak Tchebycheff systems, necessitating a precise interlacing structure of knots and assigned interpolation points or nodes for the solution of the general interpolation problem. In terms of the $B$-splines, the criterion is simply expressed: a node in the (interior) support of each $B$-spline. This whole circle of ideas was to prove a decisive force in the subsequent development of the theory and included the powerful idea of variation diminishing approximations. The author gives a careful account here and, in fact, devotes an entire chapter

[1] This paper was not without precedents as Schoenberg **[1973]** himself observes. In particular, cf. T. Grevelle **[1944]** and W. Quade and L. Collatz **[1938]**.

to Tchebycheffian splines (cf. also the study by S. Karlin, C. Micchelli, A. Pinkus and I. Schoenberg [**1976**]) and many sections to related ideas.

Part of the excitement felt by research workers at the time of the late 1960's, when a powerful synthesizing development was under way following the appearance of the book by J. Ahlberg, E. Nilson and J. Walsh [**1967**], was due to the happy coincidence of theory and application. It was known, for example, that bicubic, i.e., tensor product cubic, spline surfaces had been introduced for modelling automobile contours at the General Motors Research Laboratories in the early 1960's (cf. C. deBoor [**1962**] for a mathematical study of these surfaces), that actuaries had used splines to construct mortality tables long before 1946, and that naval architects routinely used mechanical splines, in conjunction with rotating sleeves, for curve fitting purposes. It was also known that, independently, the finite element method, introduced by R. Courant [**1943**] as a special Ritz variational projection method defined in terms of piecewise linear continuous trial functions on triangular finite elements, had undergone extensive development as a tool used for the solution of elliptic partial differential equations, particularly in the hands of structural engineers, who routinely used the local properties of the method as an effective computational device well suited to the local development of engineering models of structures. There were three principal explanations for the success of the finite element method, which had begun to complement spectral eigenfunction methods in much the same way as polynomials were being complemented by piecewise polynomials in approximation theory. These explanations, to be noted shortly, were in fact the identical ones for the success of splines, and this is not very surprising. In particular, the Ritz projection and the interpolating spline projection are concrete realizations of the same variational theory in Hilbert space and, moreover, the finite element trial spaces are really various multi-dimensional realizations of the splines. (For elaboration, cf. M. Schultz [**1973**] and G. Strang and G. Fix [**1973**].) The reasons then are as follows. First, the matrix equation $Ax = b$, encountered either in the process of operator approximation, as in the finite element method, or in the process of curve and surface fitting, involves sparse matrices $A$, in fact, often banded matrices with repeating block structure, due to the use of local bases. Thus, both direct and iterative methods of numerical linear algebra (cf. R. Varga [**1962**]), coupled with efficient storage of the sparse structure, lead to efficient algorithms for solution. The second and third reasons are perhaps not as obvious as the first, but are still readily understandable. One has to do with the rate of approximation of smooth functions by functions defined piecewise, sometimes termed a consistency property by numerical analysts, and the other with the stability of computational processes applied to solve the matrix system above whose very formulation influences the stability. The convergence question is affected not only by the trial space but also by the projection and its associated norm; we shall have more to say about this later. The stability is affected, not only by the projection in function space, but also by the basis functions used to define the matrix $A$. Even the case when $A$ is a positive definite, symmetric matrix can lead to a notoriously ill-conditioned system, and the truncations of the Hilbert matrix exemplify this. The choice of a strongly independent basis for the trial space, or the fitting space, is necessary here. The $B$-splines referred to above have this strong linear independence property, and it may be noted in passing that several delicate theoretical results associated with the estimation of best constants in norm estimation problems have been derived by use of it. A complete exposition of both the theory and computation of $B$-splines is given by the author.

We now address the variational properties satisfied by the splines. Interestingly, the earliest extremal properties of splines arose in supremum norm variational problems. Thus, J. Favard [**1937**] and N. Achieser and M. Krein [**1937**] simultaneously discovered that, among all $2\pi$-periodic functions which are $m$-fold integrals of functions in the unit ball of $L^\infty(-\pi, \pi)$, there is a function $f_0$ in this class which is farthest in the norm of $L^\infty(-\pi, \pi)$ from the trigonometric polynomials of degree $n$, and is characterized by

$$f_0^{(m)}(x) = \begin{cases} \operatorname{sgn}(\cos(n+1)x), & m \text{ even}, \\ \operatorname{sgn}(\sin(n+1)x), & m \text{ odd}. \end{cases} \tag{1}$$

A constant multiple of the periodic extension of $f_0$ to the real line was discovered by A. Kolmogorov [**1939**] as an extremal solution of the apparently unrelated Landau problem, which was solved by Kolmogorov after partial results were obtained by others. $f_0$ is, of course, a spline function with equally spaced simple knots. J. Favard [**1940**] gave an intrinsic characterization of a particular solution of the minimization problem, in the Sobolev space $W^{m,\infty}(0, 1)$ consisting of functions with $m$ essentially bounded derivatives:

$$\|s^{(m)}\|_{L^\infty(0,1)} = \inf\left\{\|f^{(m)}\|_{L^\infty(0,1)} : f(x_i) = y_i,\ i = 0, 1, \ldots, k\right\} \tag{2}$$

where $0 = x_0 < x_1 < \cdots < x_k = 1$ and $\{y_i\}_0^k$ are arbitrary real numbers. The function $s$ is a spline and it is now known (cf. C. Chui, P. Smith and J. Ward [**1976**]) that Favard's procedure yields the limit, as $p \to \infty$ (in $L^1(0, 1)$, say), of the corresponding sequence $\{s_p\}$, where $s_p$ solves the associated minimization problem in $W^{m,p}(0, 1)$, $1 < p < \infty$, when the norm of $f^{(m)}$ is computed in $L^p(0, 1)$. The functions $s_p$ are not splines except for the case $p = 2$. The extremals in each of these cases were apparently not recognized as prototypical. In any event, it was not until the 1970's that this part of the subject renewed itself. A further discussion here would be too lengthy; the reader is referred to the monograph of S. Fisher and the reviewer [**1975**] for further details. Incidentally, the function $f_0$ has been termed an *Euler spline* by I. Schoenberg [**1973**] since its components are classical Euler polynomial arcs. It is part of a much larger circle of ideas in which splines of specified power growth interpolate data of the same power growth at the integers; in the case of $f_0$ the data are elements of a bounded, indeed, binary set.

It would be fair to say that the extensive variational properties developed for splines in Hilbert space in the 1960's owed much to two major papers which appeared much earlier and which dealt with substantive characterizations which were not at first recognized as equivalent, although strong interconnections were evident. These were the papers of A. Sard [**1949**] and of M. Golomb and H. Weinberger [**1959**]. We shall describe the major ideas without technical subterfuge. Sard was attempting to construct explicit formulas approximating linear functionals such as integration, differentiation, etc., defined on smooth function classes. These formulas involved function or derivative function evaluations taken at a fixed, finite set of points specified in advance. Sard decided to select the coefficients in a formula based on fixed (possibly multiple) points in such a way that the kernel, in the Peano representation of the remainder, was minimized in $L^2$. As a reflection of this optimality, these formulas have been called *best in the sense of Sard* by other researchers. It was discovered more than a decade later that the formulas of Sard are identical to those obtained by applying the exact linear functional to the spline interpolants, a fact implicit in the paper of Golomb and Weinberger. This paper addressed a different type of optimality, viz., that of determining the centroid of a hyperellipse or hypercircle in function space. To describe this, consider an ellipsoid $\mathfrak{R} = \{f \in D_R: (Rf, Rf) \leq c\}$ in a function space such as $L^2(a, b)$, or, more generally, in any Hilbert space $H$. The (unbounded) linear operator $R$ may be taken, for example, as the differential operator $D^m$ in $L^2(a, b)$. Suppose an undetermined element $u$ is known to belong to $\mathfrak{R}$ and, simultaneously, to a hyperplane $Q \subset D_R$ intersecting $\mathfrak{R}$. Thus, in concrete terms, $Q$ might be specified as the set of functions $f \in D_R$ satisfying $f(x_i) = y_i$, $i = 0, \ldots, k$, where the partition $\{x_i\}$ of $[a, b]$ and the ordinates $\{y_i\}$ are specified. Intuitively, the best estimator of $u \in \mathfrak{R} \cap Q$ is the centroid $s$; it turns out that $s$ is identical to the element defined directly by

$$\|Rs\| = \inf\{\|Rf\| : f \in Q\}. \tag{3}$$

When $R = D^m$, $H = L^2(a, b)$ and $Q$ is defined by interpolation, $s$ is simply the natural interpolating spline in $C^{2m-2}[a, b]$ of degree $2m - 1$. The useful characterization (3) of the hypercircle centroid was not emphasized directly by Golomb and Weinberger, though they computed the spline centroids for many examples, particularly by use of reproducing kernels. In fact, the various connections evolved gradually during the 1960's. By the end of the decade, the equivalence of Sard's optimality criterion with the hypercircle method and with (3) were well established in situations of considerable generality.[2] The reader may consult the book of Sard and S. Weintraub [**1971**] for further details; also, the interesting paper of M. Golomb [**1967**]. It is noteworthy that the paper of Golomb and Weinberger [**1959**], which references the paper of Sard [**1949**], is physically contiguous to a subsequent paper of Sard [**1959**] on this subject.

The study of the general problem (3) in Hilbert space dated from the discovery by J. Holladay [**1957**] that the natural cubic interpolating spline function was an extremal of the functional $\int_a^b (f'')^2\, dx$, thus providing a smoothest interpolant if one thinks of $D^2$ as providing a linear approximation of the curvature operator. It is reasonable to inquire what occurs if the genuine strain energy, $\int \kappa^2\, ds$, of a thin elastic beam is considered as a substitute for the quadratic expression above, where $\kappa$ denotes curvature and $s$ arc length. One might expect the critical points of this functional to replicate in a precise

[2] The hypercircle inequality is, however, superior to Sard's original estimate.

manner the analog devices employed by draftsmen. Unfortunately, this problem is not amenable to the direct methods in the calculus of variations, since the infimum of the functional $\int \kappa^2 \, ds$ is zero over admissible interpolants, and this extremal value is assumed only in the trivial case when the data to be interpolated lie on a straight line segment; there is clearly no maximum of $\int \kappa^2 \, ds$. As of this writing, the interesting mathematical problem of proving the existence of critical points in the general case remains an open question, although partial results have been obtained. In particular, an application of the implicit function theorem in an appropriate function space, in a neighborhood of the ray configuration interpolant, does demonstrate the existence of critical points when the data are distributed sufficiently closely to a ray configuration. Interestingly, the linearization condition permitting the application of the implicit function theorem here is precisely the condition that cubic spline interpolation of arbitrary data is uniquely possible (cf. M. Golomb and the reviewer [**1979**]). This lends support to Schoenberg's original contention that the cubic spline was a smoothest interpolant. We note in passing that other versions of planar spline fits have been conceived which are totally unrelated to variational problems. We mention specifically the novel design by D. Knuth [**1979a**, **1979b**] of an automatic, computer-assisted type-setting process which has been adopted by the American Mathematical Society as of 1981. In private correspondence, Knuth has termed such splines, "circular cubics".

We have made mention earlier of the rate of convergence properties of the splines. Much effort has been expended in classical approximation theory in deriving rates of convergence which are characterized by the order $m$ of smoothness and the dimension $n$ of the approximating subspace; such rates typically are expressed as order expressions such as $n^{-m}$ in $\mathbf{R}^1$ and $n^{-m/N}$ in $\mathbf{R}^N$, in various function space norms. Refined versions of these estimates replace the order expression $n^{-m}$ by the $m$th modulus of smoothness evaluated at $1/n$. Moreover, it is a well-known fact that such estimates actually characterize smooth function classes, thus leading to both the direct and inverse estimation problems relating the error to the modulus of smoothness and vice versa. These problems were studied extensively for polynomials earlier in the century, beginning with the fundamental work of D. Jackson and S. Bernstein. During the last fifteen years, the analogue problems for splines have been exhaustively studied by many authors, particularly R. DeVore. This work is reported in considerable detail in the book under review. In particular, the fixed and variable knot problem in one dimension as well as the tensor product fixed knot case are discussed.

It was apparently the Aachen group in approximation theory, under the leadership of P. Butzer, which recognized the decisive role that the theory of intermediate spaces could play in the resolution of the direct and inverse problems of approximation theory in a very general setting. The recognition that the Peetre $K$-functional is essentially a modulus of smoothness greatly facilitated the multi-dimensional theory by permitting the application of intermediate and interpolation space theory, particularly the characterization of the Lorentz spaces as Lebesgue interpolation spaces and the Besov spaces as Sobolev interpolation spaces. It is striking that the results presented in the book of P. Butzer and H. Berens [1967] came so soon after the theory of intermediate spaces as introduced and developed just a few years earlier by A. Calderon, J. Lions, J. Peetre and others. We can only give the reader a morsel of this theory, sometimes termed a saturation theory, which attempts to straddle the direct and inverse problems by precisely identifying the comprehensive nontrivial class for which a given approximation order is exactly maximal. Thus, the modulus of smoothness may be defined in terms of iterated translation, the infinitesimal generator of such translation is differentiation, and saturation (best possible) phenomena may be expressed in terms of the domain of such an iterated generator, which domain may coincide with an interpolation space.

A concept for characterizing the best approximation of compact or near compact classes by subspaces of dimension $n$ was introduced by A. Kolmogorov [1936]. Given a compact set $A$ in a normed linear space $X$ and a subspace $M$ of dimension $n$, we define the dispersion of $A$ from $M$ by

$$E(A, M) = \sup_{y \in A} \inf_{x \in M} \|y - x\|, \tag{4i}$$

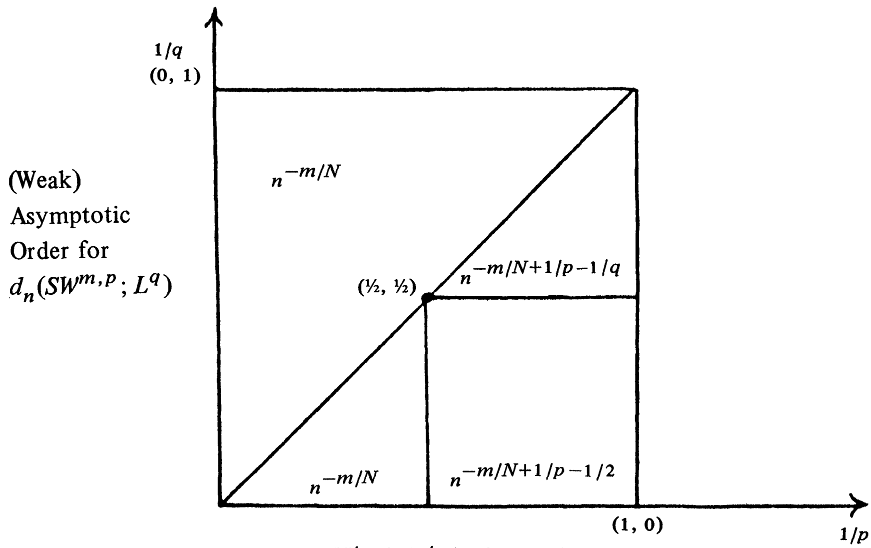
and the $n$-width of $A$ in $X$ by

$$d_n(A, X) = \inf\{E(A, M) \colon \dim M \leqslant n\}. \tag{4ii}$$

The reason for mentioning this important concept here is the interesting interface which it enjoys with splines. In the most obvious relation, we can ascribe optimal order approximation properties to splines defining subspaces $S_n$ of dimension $n$, in an asymptotic sense, if

$$E(A, S_n)/d_n(A, X) \leqslant c, \qquad n \to \infty. \tag{5}$$

Typical choices of $A$ and $X$ might be the unit ball of $W^{m,p}(\Omega)$ for $A$ and $L^q(\Omega)$ for $X$, where the relation among $m$, $p$, and $q$ is such that the embedding $W^{m,p} \to L^q$ is compact; of course, the Euclidean dimension $N$ of the bounded set $\Omega$ enters here through the Sobolev embedding theorem. The derivation of a formula such as (5) requires a knowledge of (upper and) lower asymptotic bounds for $d_n(A, X)$. For the special case cited above, these have been determined by S. Kašin [1977] and are displayed in the following tableau. In the determination of the lower bounds, splines play a decisive role via an application of the Borsuk antipodal theorem to the function $f(u) = w \in M$ if $\|u - w\| = \inf\{\|u - v\| \colon v \in M\}$, $u \in \partial A$, in a finite dimensional spline subspace $M$ of dimension $n + 1$. In certain cases degenerate splines provide upper bounds via a local application of Sobolev's integral representation formula, e.g., in the case $1 \leqslant p \leqslant q \leqslant 2$. However, as the author notes in §6.6, this case and the case $1 \leqslant q \leqslant p \leqslant \infty$ are the only cases in which (5) holds. A greater compatibility exists between the splines and the so-called linear $n$-widths, defined via linear operators. There also exist other sharply defined relationships between splines and $n$-widths. Thus, for example, the ellipsoid $\mathfrak{R}$ described earlier has its finite widths equal to the reciprocal square root of the eigenvalues of $R^*R$ (cf. the reviewer [1967]); optimal spaces assuming the $n$-width consist not only of eigenfunctions but, in concrete cases, splines as well (cf. A. Melkman and C. Micchelli [**1976**]). In $L^\infty$, the splines also appear as extremals in the $n$-width characterization and, in addition, there is a remarkable result of V. Tihomirov [**1969**] which relates the $n$-widths to the oscillations of a family of splines in a natural way.




$m - N/p + N/q > 0$, *and*
$m - N/2 > 0$ if $2 \leqslant p < q$,
$m - N/p > 0$ if $p < 2 < q$.

The reader will recall that the Fourier partial sum projections form an unbounded sequence of operators on the $2\pi$ periodic subspace of $C[-\pi, \pi]$, which explains why there are continuous functions with nonuniformly convergent Fourier series. Similarly, one may inquire whether the least squares spline projections, defined in $L^2$, constitute a bounded sequence on $L^\infty$ for reasonable partition sequences. Here the answer is an affirmative one, and there are related results for finite elements (cf. C. deBoor [**1976**] and J. Douglas, T. Dupont and L. Wahlbin [**1975**]). This illustrates the flexibility of piecewise polynomials with respect to classes of analytic functions, which often behave inflexibly. This inflexibility accounts for several undesirable properties of polynomials. One of the most famous examples, given by the author, is the divergence, on (part of) the interval $[-5, 5]$, of the Lagrange interpolation polynomials defined on uniform partitions for the real analytic function $f(x) = (1 + x^2)^{-1}$. Splines, of course, behave magnificently and the interpolants converge uniformly for any continuous function. There is a final comment worth making here. If one is willing to forsake the linear structure of fixed knots for the manifold structure of variable knot spline approximation, an even greater flexibility is obtained. This was first noticed for piecewise constants by J. Kahane [**1961**] and subsequently investigated by many others. Of interest here is the fact that the Hölder function $x^\sigma$, $0 < \sigma < 1$, can be approximated uniformly to order $k^{-m}$ by splines of degree $m$ with $k$ free knots on $[0, 1]$ while the optimal order for fixed knot approximation is $k^{-\sigma}$. The saturation theory here is not complete.

We turn now to some final direct comments about the book. This concisely written 553-page book is multiple-tiered. The author has not attempted to write a textbook as such, though he notes that, "With a judicious choice of material, the book can be used for a one-semester introduction to splines". In this case, the prerequisites would be calculus and linear algebra. However, there is considerable material to challenge the sophisticated mathematician. The author apparently intends to follow this volume with others. In fact, his approach has been single-mindedly constructive and algorithmic, and many of the variational properties mentioned above will not be found here. For the novice, a useful companion volume would be C. deBoor's compact text [**1978**]. A complementary readable account is the book edited by T. Greville [**1969**].

The book is exceptionally well referenced and annotated. The visual displays are apt and helpful, and examples and remarks are generously provided. Complicated ideas are presented clearly. The reader should be warned, however, that there is a great deal here, and even this is far from the entire story. Cue, volume 2!

## References

1. N. Achieser and M. Krein [**1937**], *On the best approximation of periodic functions*, Dokl. Akad. Nauk SSSR **15**, 107–112.

2. J. Ahlberg, E. Nilson and J. Walsh [**1967**], *The theory of splines and their applications*, Academic Press, New York.

3. C. deBoor [**1962**], *Bicubic spline interpolation*, J. Math. Phys. **12**, 747–749.

4. ______ [**1976**], *On local linear functionals which vanish at all B-splines but one*, Theory of Approximation, with Applications (Law and Sahney, eds.), Academic Press, New York, pp. 120–145.

5. ______ [**1978**], *A practical guide to splines*, Springer-Verlag, New York.

6. P. Butzer and H. Berens [**1967**], *Semi-groups of operators and approximation*, Springer-Verlag, Berlin.

7. E. Cheney [**1966**], *Introduction to approximation theory*, McGraw-Hill, New York.

8. C. Chui, P. Smith and J. Ward [**1976**], *Favard's solution is the limit of $W^{k,p}$-splines*, Trans. Amer. Math. Soc. **220**, 299–305.

9. R. Courant [**1943**], *Variational methods for the solution of problems of equilibrium and vibration*, Bull. Amer. Math. Soc. **49**, 1–23.

10. P. Davis [**1963**], *Interpolation and approximation*, Blaisdell, New York.

11. J. Douglas, T. Dupont and L. Wahlbin [**1975**], *Optimal $L^\infty$ error estimates for Galerkin approximations to solutions of two-point boundary value problems*, Math. Comp. **29**, 475–483.

12. J. Favard [**1937**], *Sur les meilleure procédés d'approximation de certaines classes des fonctions par des polynômes trigonométriques*, Bull. Sci. Math. **61**, 209–224; 243–256.

13. ______ [**1940**], *Sur l'interpolation*, J. Math. Pures Appl. **19**, 281–306.

14. S. Fisher and J. Jerome [**1975**], *Minimum norm extremals in function spaces*, Lecture Notes in Math., no. 479, Springer-Verlag, Berlin.

15. M. Golomb [**1962**], *Lectures on theory of approximation*, Argonne National Laboratory.

16. ______ [**1967**], *Splines, n-widths and optimal approximations*, MRC Tech. Rpt. 784, Mathematics Research Center, Madison, Wisconsin.

17. M. Golomb and J. Jerome [**1979**], *Equilibria of the curvature functional and manifolds of nonlinear interpolating spline curves*, MRC Tech. Rpt. 2024, Mathematics Research Center, Madison, Wisconsin. Also, SIAM J. Math. Anal., 1982.

18. M. Golomb and H. Weinberger [**1959**], *Optimal approximation and error bounds*, On Numerical Approximation (R. Langer, ed.), University of Wisconsin Press, pp. 117–190.

19. T. Greville [**1944**], *The general theory of osculatory interpolation*, Trans. Actuar. Soc. Amer. **45**, 202–265.

20. ______ [**1969**], *Theory and applications of spline functions*, Academic Press, New York.

21. J. Holladay [**1957**], *A smoothest curve approximation*, Math Tables Aids Computation **11**, 233–243.

22. J. Jerome [**1967**], *On the $L_2$ n-width of certain classes of functions of several variables*, J. Math. Anal. Appl. **20**, 110–122.

23. J. Kahane [**1961**], *Teoria constructiva de functiones*, University of Buenos Aires.

24. S. Karlin [**1968**], *Total positivity*, Stanford University Press.

25. S. Karlin, C. Micchelli, A. Pinkus and I. Schoenberg [**1976**], *Studies in spline functions and approximation theory*, Academic Press, New York.

26. S. Karlin and W. Studden [**1966**], *Tchebycheff systems with applications in analysis and statistics*, Wiley Interscience, New York.

27. B. Kašin [**1977**], *The widths of certain finite dimensional sets and classes of smooth functions*, Izv. Akad. Nauk. SSSR Ser. Mat. **11**, 334–351.

28. D. Knuth [**1979a**], *Mathematical typography*, Bull. Amer. Math. Soc. (N.S.) **1**, 337–372.

29. ______ [**1979b**], *Tex and metafont*, (Part 3, pp. 20–21), Digital Press, North Billerica, Mass.

30. A. Kolmogorov [**1936**], *Annäherung von Funktionen einer gegebenen Funktionklasse*, Ann. of Math. **31**, 107–111.

31. ______ [**1939**], *On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval*, Amer. Math. Soc. Trans. Series 1, 2 (1962), 233–243.

32. G. Lorentz [**1966**], *Approximation of functions*, Holt, Rinehart and Winston, New York.

33. G. Meinardus [**1967**], *Approximation of functions: Theory and numerical methods*, Springer-Verlag, Heidelberg.

34. A. Melkman and C. Micchelli [**1976**], *On non-uniqueness of optimal subspaces for $L^2$ n-width*, IBM Res. Rpt. RC 6113, Yorktown Hts., N. Y.

35. W. Quade and L. Collatz [**1938**], *Zur Interpolationstheorie der reelen periodischen Funktionen*, Akad. Wiss. **30**, 383–429.

36. J. Rice [**1964**], *The approximation of functions*, Vol. 1, Linear Theory, Addison-Wesley, Reading, Mass.

37. A. Sard [**1949**], *Best approximate integration formulas, best approximation formulas*, Amer. J. Math. **71**, 80–91.

38. ______ [**1959**], *The rationale of approximation*, On Numerical Approximation (R. Langer, ed.), University of Wisconsin Press, pp. 191–207.

39. ______ [**1963**], *Linear approximation*, Math. Surveys no. 9, Amer. Math. Soc., Providence, R. I.

40. A. Sard and S. Weintraub [**1971**], *A book of splines*, Wiley, New York.

41. I. Schoenberg [**1946**], *Contributions to the problem of approximation of equidistant data by analytic functions*, Quart. Appl. Math. **4**, 45–99 (Part A), 112–141 (Part B).

42. ______ [**1973**], *Cardinal spline interpolation*, CBMS 12, SIAM, Philadelphia.

43. M. Schultz [**1973**], *Spline analysis*, Prentice-Hall, Englewood Cliffs, N. J.

44. G. Strang and G. Fix [**1973**], *An analysis of the finite element method*, Prentice-Hall, Englewood Cliffs, N. J.

45. V. Tihomirov [**1969**], *Best methods of approximation and interpolation of differentiable functions in the space $C[-1, 1]$*, Mat. Sb. **80**, 290–304.

46. A. Timan [**1963**], *Theory of approximation of functions of a real variable*, McMillan, New York.

47. R. Varga [**1962**], *Matrix iterative analysis*, Prentice-Hall, Englewood Cliffs, N. J.

JOSEPH W. JEROME