mathematical development). These provide surveys with pleasant points of view and could be used as textual material in a number of contexts (e.g. undergraduate applied mathematics students in some cases) where there would be definite benefits. Additional less standard, more modern types of applications (such as the analysis of pair-wise ratio comparisons to establish cardinal rankings or in graph theory and combinatorics [**4**]) would also have been welcome in this part of the book.

Any treatment of nonnegative matrices would be highly author-dependent, and the present work, which uniquely reflects the authors' tastes, makes a definite contribution. However, there are other possible developments of this subject and a further book in the field should not be precluded. An alternative would have been a work less general but more primarily, deeply and directly about nonnegative matrices and more internally developed. Such might be more useful to the casual applier and also allow fuller internal development of the fancier topics for the mathematician. Perhaps that will come sometime.

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Minimal factorization of matrix and operator functions, by H. Bart, I. Gohberg and M. A. Kaashoek, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1979, v + 227 pp., $17.50.

This is a monograph on system theory and a branch of operator theory known as operator models. It makes basic contributions to both subjects and should be widely read by people in both areas. System theory is a branch of theoretical engineering while model theory evolved as pure mathematics. In the early 1970's it was shown that the two independent subjects are fundamentally equivalent. "Equivalent" is a blurry word to use in comparing two large highly developed fields, and there were discrepancies. Roughly speaking, system theory was primarily finite dimensional, while model theory was well developed in infinite dimensions. On the other hand, model theory worked smoothly for what amounts to energy conserving situations but became involved otherwise, while system theory needed no such restrictions; so structurally speaking, system theory was more general than operator model theory. This is the case partly because model theory is a theory of unitary equivalence, while system theory allows equivalence up to similarities. Once people saw the connection between systems and models, a general theory containing both developed quickly.

This unified theory contained only the most nebulous structure of the two subjects. Since the motivation behind the subjects was entirely different, specific results in the two areas focused on very different issues. For example, there was great attention to least squares quadratic control among system theorists and nothing analogous in operator theory. On the other hand, the main theme in model theory went like this: Given an operator A, one associated to it a matrix (or operator) valued function $\theta_A(z)$ ($z \in \mathbf{C}$), called its characteristic function. Factorizations of θ_A correspond to invariant subspace decompositions of A. This correspondence between factorizations and invariant subspaces was one of what I consider to be the three cornerstones on which model theory rests. The very surprising thing is that as of 1970 there was no counterpart of this basic fact in system theory. Various people (under the influence of model theory) looked at implications of this principle for particular systems through the mid-seventies. However, the general principle, which cut to the core of the matter, was not complete until Bart, Kaashoek, Gohberg, and Van Dooren added more detail to a result announced by Sahnovic (Doklady '76). The monograph under review is essentially a description of the many ramifications of this simple, beautiful principle for system theory, operator model theory, and for the study of factorization of matrix rational functions. I certainly feel that it is a substantial contribution to each of these areas.

Recall that a system is simply a set of first order differential equations

$$(*) \qquad \begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \qquad x(0) = 0, \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

Here one thinks of the driving term $u(t)$ as the input to a box at time t, $y(t)$ as the output of the box, and $x(t)$ as the internal state of the box. Typically u and y take values in a vector space $\mathbf{C}^k$ of much lower dimension than the image space $\mathbf{C}^n$ for x. In studying this system one usually works with Fourier transforms and easily finds the relation between input and output to be $\hat{y}(p) = T(p)\hat{u}(p)$, where $T(p) = D + C(p - A)^{-1}B$. Usual conventions are to call the quadruple of matrices $[A, B, C, D]$ a *system* and T its *transfer function*. There are several branches of system theory. The largest one is control theory: How do we select the input $u(t)$ to make the state $x(t)$ do what we want? This is not the concern of the book under review so we move on. Another branch of system theory is realizability theory: Given some input-output behavior how do we find a system which has it? For example, in circuit design one typically is given certain specs and must build a circuit which meets those specs; to wit, given $T(p)$ build a system which has transfer function equal to $T(p)$.

One way to proceed is to take the given function $T(p)$ and decompose it into much simpler functions $T_j(p)$. If $T_j(p)$ is very simple, then it is very easy to find a system $\mathfrak{R}_j$ (or circuit) with transfer function $T_j(p)$. If the relationship between T and its constituent T_j's is not complicated, then there is probably an easy way to connect the $\mathfrak{R}_j$'s together to get a system $\mathfrak{R}$ whose transfer function is T. Thus various decompositions of T play an important role in "realizing" the function T_j. Whereas factorization $T = T_1T_2$ is a basic type of decomposition (used, for example, in the Darlington approach to filter design), it is natural to ask what factorization of T means in terms of a system $[A, B, C, D]$ whose transfer function is T. In other words, to what does factorization of T correspond in terms of the associated system $(*)$ of differential equations?

The factorization principle describes this correspondence. It is simple enough that it can be stated immediately. It says that for a "minimal" system with invertible D,

THEOREM. *Each "minimal" factorization $T_2(p)T_1(p) = T(p)$ corresponds to a decomposition $\mathbf{C}^n = \mathbb{S}_1 + \mathbb{S}_2$ of $\mathbf{C}^n$ into nonintersecting subspaces $\mathbb{S}_1$ and $\mathbb{S}_2$ with the invariance properties*

$$A\mathbb{S}_1 \subset \mathbb{S}_1, \qquad [A - BD^{-1}C]\mathbb{S}_2 \subset \mathbb{S}_2.$$

The converse is also true.

In the rational case, a factorization being *minimal* roughly means that there is no zero-pole cancellation between the factors.

The correspondence in the theorem is simple and explicit. It has the appealing property that the poles of T_1 lie on the opposite side of a contour C from those of T_2 if and only if the spectrum of $A|_{\mathbb{S}_1}$ is on the opposite side of C from spectrum $A - BD^{-1}C|_{\mathbb{S}_2}$. So the theorem is an excellent tool for studying "spectral" factorizations of a given T, namely, factorizations whose factors have poles in prescribed regions.

The first chapter of the book is especially elegant. It presents the factorization principle, the relation to spectral factorization, and applies this to various classical operator models such as the Livsic-Brodskii model and the Krein model (a variant on the Nagy-Foiaş model), as well as to more recent polynomic models. By the end of the chapter one has a unified approach to factorization in all of these models.

Besides factorization versus invariant subspaces, there remain two cornerstones of model theory, system theory, and the newer unified theory. The first starts with a matrix valued analytic function T and tells how to construct a system having T as its transfer function. The second says that the transfer function determines a "minimal" system up to "similarity". Chapters II and III treat these issues thoroughly and bring some new ideas which add more detail. Thus, the first three chapters together give a solid introduction to the basics of model and system theory. The remaining two-thirds of the book concentrates on applications of Theorem I.

A major issue ($2\frac{1}{2}$ chapters worth) is the stability of factors T_1 and T_2 with respect to perturbations of T. Conversion to an invariant subspace problem yields satisfactory results on the issue of which T have stable factorizations.

An intriguing observation is an explicit correspondence between factorizations of T and the solutions of an algebraic Ricatti equation. They use this to study stable solutions to the Ricatti equations. Related relationships I have seen (in the engineering literature) are in comparatively special circumstances.

Most of the material in the book fits easily into infinite dimensional space and that is where it is done. The authors are consequently able to study certain integro-differential equations. In particular their methods apply to the transport equation (of nuclear physics) and a chapter is devoted to this. This approach to the transport equation has proved to be valuable and the interested reader should see a forthcoming book on transport equations to appear in the same Birkhäuser series.

There are many other nice ideas which cannot be mentioned in a brief review. In summary, the first third of the book sets out principles of model and system theory of such general interest that it could serve as an introduction to many readers. It does not give physical motivation or many references to the systems literature, so the beginner would want a more engineering oriented supplement (e.g. T. Kailath's book). Also to fill in more model theory, one could see either the definitive book of Nagy and Foiaş or the more informal account of the Nagy-Foiaş theory, by R. G. Douglas, which is contained in the volume of the MAA studies series which C. Pearcy edited. Also there is Brodskii's book. The remainder of the book is also accessible with little background and contains much fine mathematics.

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Spline functions: *Basic theory*, by Larry L. Schumaker, Wiley, New York, 1981, xiv + 553 pp., $42.50.

A polynomial spline function results from splicing polynomial arcs in such a way that the resulting function is sufficiently smooth. In more precise language, a polynomial spline function of degree $k \geqslant 0$ is a real function defined by piecewise polynomial components of degree $\leqslant k$ whose derivatives through order $k - 1$ are continuous. The juncture points are commonly referred to as (simple) *knots* in the literature. Central to the study of these functions is the class of *minimal support splines* or *B-splines*. It is found that the smallest possible number of knots of a spline of degree k whose support is a compact interval in the interior of its domain is $k + 2$. Such splines are uniquely determined up to constant multiples. They are ideal basis functions and can be calculated recursively by formulas which express a B-spline of a given degree k as a convex linear combination of two B-splines of degree $k - 1$.