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BOOK REVIEWS

Elliptic pseudo-differential operators–an abstract theory, by H. O. Cordes, Lecture Notes in Math., vol. 756, Springer-Verlag, Berlin and New York, 1979, 331 pp., $18.00.

Pseudodifferential operators (often called ψDOs) are generalizations of differential operators, and they arose to treat problems in partial differential equations. One common characterization of a ψDO is as an operator of the form

$$Pu(x) = \int p(x, \xi) e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi \qquad (x, \xi \in \mathbf{R}^n) \tag{1}$$

where $\hat{u}(\xi) = (2\pi)^{-n} \int e^{-ix\cdot\xi} u(x)\, dx$ is the Fourier transform of u. Formula (1) defines a differential operator in case $p(x, \xi)$ is a polynomial in ξ. More generally, $p(x, \xi)$ can belong to a symbol class, such as the symbol class of Hörmander

$$p(x, \xi) \in S^m_{\rho,\delta} \Leftrightarrow \left| D_x^\beta D_\xi^\alpha p(x, \xi) \right| \leqslant C_{\alpha\beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \tag{2}$$

or other classes, e.g., due to Beals and Fefferman **[1]**, **[2]**, or Hörmander **[11]**. Such operators captured the attention of many mathematicians, not necessarily primarily interested in partial differential equations, in the mid '60s, because of the role they played in the proof of the Atiyah-Singer index theorem, particularly in the production of families of operators known to be Fredholm, connecting together two elliptic differential operators with homotopic principal symbol, to prove the index of such an operator depends only on the homotopy class of its principal symbol.

An operator on a Hilbert space H is Fredholm if and only if it is invertible modulo the algebra $\mathcal{K}$ of compact operators. Thus, given a *-algebra $\mathfrak{A}_0$ of ψDOs, say of the form (1) with $p(x, \xi)$ perhaps belonging to a subclass of symbols of the form (2), to study Fredholm properties of elements of $\mathfrak{A}_0$ it is natural to look at the quotient algebra $\mathfrak{A}/\mathcal{K}$, where $\mathfrak{A}$ is the L^2-operator norm closure of $\mathfrak{A}_0$, perhaps with $\mathcal{K}$ thrown in. If $\mathfrak{A}$ acts irreducibly on H and contains one compact operator, as is often the case, it is not hard to show $\mathfrak{A}$ contains $\mathcal{K}$ (see **[15**, p. 192]). If $\mathfrak{A}_0$ consists of operators of the form (1), (2), with $m = 0$, $\rho = 1$, $\delta = 0$, and $p(x, \xi)$ well behaved at infinity, then commutators $[P, Q] = PQ - QP$ of elements of $\mathfrak{A}_0$ are compact and hence $\mathfrak{A}/\mathcal{K}$ is a commutative C^* algebra. The same holds if $\mathfrak{A}_0$ is the algebra of "classical" pseudodifferential operators of order zero on a compact manifold X, without boundary. Thus, $\mathfrak{A}/\mathcal{K}$ is isomorphic to $C(M)$, the algebra of continuous complex valued functions on a compact Hausdorff space M, which in the case of the last mentioned example turns out to be $S^*(X)$, the cosphere bundle of X. An element A of $\mathfrak{A}_0$ thus gives rise to a function $\sigma(A)$ on M, and A is Fredholm if and only if $\sigma(A)$ is nowhere vanishing on M. (To see this, one needs to remark that $A \in \mathfrak{A}$ is invertible in $\mathfrak{A}/\mathcal{K}$ if and only if it is invertible in $\mathcal{L}/\mathcal{K}$, where $\mathcal{L}$ is the algebra of bounded operators on H.) The book under review is concerned with the C^* algebra approach to Fredholm properties of elliptic operators.

For the purpose of studying elliptic operators, it is convenient to define classes of ψDOs $\mathfrak{A}_0$ by specifying some generating subalgebras. For example, let $\mathfrak{A}_0$ be the algebra generated by the algebras

$$\mathcal{M}_1 = \{Tu(x) = p(x)u(x): p(x) \text{ satisfies } (5)\} \tag{3}$$

and

$$\mathcal{M}_2 = \left\{Tu(x) = p(D)u(x) = \int e^{ix\cdot\xi}p(\xi)\hat{u}(\xi)\,d\xi: p(\xi) \text{ satisfies } (5)\right\} \tag{4}$$

where, in either case, $p(\lambda)$ is smooth and

$$p(\lambda) \sim \sum_{j \leq 0} p_j(\lambda) \quad \text{as } |\lambda| \to \infty, \tag{5}$$

where $p_j(\lambda)$ is homogeneous of degree j in $\lambda \in \mathbf{R}^n$. Clearly the norm closures $\overline{\mathcal{M}}_1$ and $\overline{\mathcal{M}}_2$ are both commutative C^* algebras, with

$$\overline{\mathcal{M}}_1 \approx C(\mathbf{B}^n) \approx \overline{\mathcal{M}}_2 \tag{6}$$

where $\mathbf{B}^n$ is a compactification of $\mathbf{R}^n$, homeomorphic to the unit ball. Indeed, the dominant term in (5), for large $|\lambda|$, is

$$p_0(\lambda) = p_0(\omega), \qquad \lambda = r\omega, \quad \omega \in S^{n-1},$$

so we see that S^{n-1} is naturally attached to R^n as the set of "points at infinity" to give (6). Given (3)–(5), one can show that commutators of elements of $\mathcal{M}_1$ with elements of $\mathcal{M}_2$ are compact, and hence $\mathfrak{A}/\mathcal{K}$ is commutative in this case. Generally, if $\mathfrak{A}$ is generated by two commutative C^* algebras $\overline{\mathcal{M}}_j$ with maximal ideal spaces M_j, and if $\mathfrak{C}$ denotes the commutator ideal of $\mathfrak{A}$, the natural maps $\overline{\mathcal{M}}_j \to \mathfrak{A}/\mathfrak{C}$ induce a homeomorphism of the maximal ideal space M of $\mathfrak{A}/\mathfrak{C}$ ($\mathfrak{A}/\mathfrak{C} \approx C(M)$) onto a closed subspace of $M_1 \times M_2$,

$$M \subset M_1 \times M_2. \tag{7}$$

In the case of the algebra $\mathfrak{A}$ arising from (3)–(5), we wish to determine the maximal ideal space of M of $\mathfrak{A}/\mathcal{K}$ explicitly as a subset of $\mathbf{B}^n_x \times \mathbf{B}^n_\xi$, which is a compactification of $\mathbf{R}^n_x \times \mathbf{R}^n_\xi$. Indeed, it is easy to produce lots of compact operators $p(x)q(D)$ with p and q compactly supported, and deduce that $M \cap (\mathbf{R}^n_x \times \mathbf{R}^n_\xi) = \varnothing$, i.e.,

$$M \subset \partial(\mathbf{B}^n_x \times \mathbf{B}^n_\xi) = \mathbf{B}^n_x \times \mathbf{B}^n_\xi \setminus \mathbf{R}^n_x \times \mathbf{R}^n_\xi. \tag{8}$$

One identifies M by proving *equality* in (8). In fact, consider the part of M lying over $\mathbf{R}^n_x \subset \mathbf{B}^n_x$. Because translations and rotations on $\mathbf{R}^n$ give isometries on $L^2(\mathbf{R}^n)$ which yield automorphisms of $\mathcal{M}_1$ and $\mathcal{M}_2$, and hence of $\mathfrak{A}$, we can deduce from the existence of one point lying over $\mathbf{R}^n_x$ (which follows because no nonzero element of $\mathcal{M}_1$ is compact) that we have everything in (8) which lies over $\mathbf{R}_x^n$ belonging to M. Consequently one obtains $\mathbf{R}_x^n \times \partial\mathbf{B}_\xi^n \subset M$. Since the Fourier transform induces an automorphism of M that interchanges the roles of x and ξ, we have $\partial\mathbf{B}_x^n \times \mathbf{R}_\xi^n \subset M$, which gives equality in (8).

Thus one has a particularly painless way to determine which elements of $\mathfrak{A}$, generated by (3)–(5) above, are Fredholm on $L^2(\mathbf{R}^n)$. The book of Cordes examines larger classes of ψDOs, where such simple symmetry considerations need to be replaced, and also considers algebras acting on Sobolev spaces. The book proceeds to a second major topic: the study of Fredholm properties of elliptic operators on manifolds with boundary, where the set-up is a bit more elaborate than above. In fact, let Ω be a smooth compact manifold with boundary X. Define $(1 - \Delta_d)^{-1}$: $L^2(\Omega) \to H^2(\Omega)$ and $(1 - \Delta_n)^{-1}$: $L^2(\Omega) \to H^2(\Omega)$ to be the solution operators to

$$(1 - \Delta)u = f \quad \text{on } \Omega$$

with, respectively, Dirichlet boundary conditions $u|_X = 0$, or Neumann boundary conditions $(\partial/\partial\nu)u|_X = 0$. Let $\Lambda_d = (1 - \Delta_d)^{-1/2}$, $\Lambda_n = (1 - \Delta_n)^{-1/2}$, so Λ_d, Λ_n: $L^2(\Omega) \to H^1(\Omega)$, where $H^k(\Omega)$ is the Sobolev space of $u \in L^2(\Omega)$ such that $D^\alpha u \in L^2(\Omega)$ for $|\alpha| \leqslant k$. Define the algebra $\mathfrak{A}$ on $L^2(\Omega)$ to be the norm closure of the algebra generated by

$$Z\Lambda_d, Z\Lambda_n, p(x) \tag{9}$$

where Z runs over the set of smooth vector fields on $\bar{\Omega}$, $p(x) \in C^\infty(\bar{\Omega})$. This time, commutators of elements of $\mathfrak{A}$ are not compact. However, if $\mathfrak{C}$ denotes the commutator ideal, one still has

$$\mathfrak{A}/\mathfrak{C} \approx C(M_1) \tag{10}$$

where M_1 is the boundary of an appropriate compactification of $T^*(\Omega)$. Furthermore, one has

$$\mathfrak{C}/\mathcal{K} \approx C(M_2, \tilde{\mathcal{K}}) \tag{11}$$

where $M_2 = S^*(X)$ and $\tilde{\mathcal{K}}$ is the algebra of compact operators on an auxiliary Hilbert space H'. Thus the problem of determining if an element $A \in \mathfrak{A}$ is Fredholm is attacked in two stages. First, its image $\sigma_1(A) \in \mathfrak{A}/\mathfrak{C} \approx C(M_1)$ must be invertible for A to be Fredholm, i.e., $\sigma_1(A)$ must be nowhere vanishing on M_1. If this is true, pick an element B which is an inverse mod $\mathfrak{C}$, so $AB = I + C$, $C \in \mathfrak{C}$. To check invertibility mod $\mathcal{K}$ of $I + C$, consider the image of C in $\mathfrak{C}/\mathcal{K}$, $\sigma_2(C) \in \mathfrak{C}/\mathcal{K} \approx C(M_2, \tilde{\mathcal{K}})$. We demand that, for each $\zeta \in M_2$, $1 + \sigma_2(C)(\zeta)$ be an invertible operator on H'. From the composition series (10), (11), it follows that $\mathfrak{p} = \mathfrak{A}/\mathcal{K}$ is a separable, type I, C^* algebra.

One can regard the algebra of ψDOs as a particular case of a C^* algebra $\mathfrak{A}$ in $\mathfrak{L}$, containing the algebra $\mathcal{K}$ of compact operators, such that $\mathfrak{A}/\mathcal{K}$ is isomorphic to a particular commutative C^* algebra $C(M)$, or more generally such that $\mathfrak{A}/\mathcal{K}$ is isomorphic to some given separable C^* algebra $\mathfrak{B}$. If one identifies such algebras when they are conjugate, one obtains a set denoted $\mathrm{Ext}(M)$ in the former case, or more generally, $\mathrm{Ext}(\mathfrak{B})$, as described in [7]. One has a natural addition defined, and it is a deep result that $\mathrm{Ext}(M)$ is a group. The monograph [7] sketches some important topological properties of $\mathrm{Ext}(M)$ and connections with the index theorem. In the case $M = S^*(X)$ for a compact boundaryless manifold X, there is an additional structure on M, a contact structure. This also appears in more general contexts, such as the study of Toeplitz operators on a strictly pseudoconvex domain; see Boutet de Monvel [3]. An abstract study of this situation, suggested by Singer [16], has been developed less fully than the general study of Ext(M). Also there are results for Ext($\mathfrak{B}$) for a separable type I C^* algebra $\mathfrak{B}$, not necessarily commutative; see [7] for references. As indicated above, natural algebras of pseudodifferential operators on manifolds with boundary give rise to examples of elements of Ext($\mathfrak{B}$), for $\mathfrak{B} = \mathfrak{p} = \mathfrak{A}/\mathfrak{K}$.

Various techniques have been developed for the theory of pseudodifferential operators, and there have been several monographs giving special emphasis to specific techniques. Let us particularly mention methods from harmonic analysis, different aspects of which are presented in Coifman and Meyer [4] and in Nagel and Stein [14], and methods from symplectic geometry, which are particularly incisive in the generalization from pseudodifferential operators to Fourier integral operators, exposed in Duistermaat [8] and in Guillemin and Sternberg [9].

Furthermore, it is worth mentioning that the use of elementary operator theory has played a striking role in proving some important inequalities for pseudodifferential operators, the Calderón-Vaillancourt theorems. The use of the Cotlar-Stein lemma on sums of almost orthogonal operators is well known. Cordes [5] and Kato [13] have proved these theorems by observing that if $U(y)$, $y \in Y$, is a square integrable family of unitary operators in the sense that

$$\int_y |(U(y)f, g)|^2 \, dy \leqslant C\|f\|^2 \, \|g\|^2,$$

and if

$$B = \int_Y b(y)\, U(y)^* G U(y) \, dy$$

with $b \in L^\infty(Y)$, G trace class, then

$$\|B\| \leqslant C'\|b\|_{L^\infty}\|G\|_{\mathrm{tr}}.$$

One exploits this with $y = (z, \xi) \in T^*(\mathbf{R}^n)$, $U(y) = e^{iz\cdot D}e^{i\xi\cdot X}$, where $z \cdot D = z_1D_1 + \cdots + z_nD_n$, $D_j = (1/i)(\partial/\partial x_j)$, $\xi \cdot X = \xi_1x_1 + \cdots + \xi_nx_n$. Thus estimates on the operator norm of B result from the square integrability of the Stone-von Neumann representation of the Heisenberg group, a fact also emphasized by Howe [12].

As mentioned, the book of Cordes emphasizes a C^* algebra viewpoint. The book is written to be accessible to beginners. Appendices giving an efficient sketch of some basic C^* algebra theory are included, and a couple of introductory chapters sketch basic distribution theory and discuss some of the singular distributions which arise as kernels of classical pseudodifferential operators.

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Dimension theory, by Ryszard Engelking, North-Holland Mathematical Library, Vol. 19, North-Holland Publishing Company, Amsterdam and New York; Polish Scientific Publishers, Warsaw, 1978, x + 314 pp., $44.50.

Geometry lays claim to being the oldest mathematical discipline. The notion of dimension is fundamental to geometry, but was without adequate rigorous underpinnings until the twentieth century. The early work of dimension theorists culminated in *Dimension theory* by W. Hurewicz and H. Wallman in 1941. Here the intuitive concepts of dimension were given precise definition and a complete theory for finite-dimensional separable metric spaces was given in an elegant and succinct form. There were many areas which remained to be investigated. One could argue that there should exist a comparable theory for general metric spaces. Within a few years such a theory was mapped out. J. Nagata's book, *Modern dimension theory*, relates the essential features of this theory. The intervening years have given us only minor embellishments. Dimension theory for nonmetrizable spaces is at the present time in a very unsatisfactory state, but for a different reason than in the past. Today we know that a satisfactory theory does not exist. Even compact spaces have proven perverse. Only Lebesgue covering dimension has