BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 3, May 1981

LINEAR GROUPS OF FINITE COHOMOLOGICAL DIMENSION

BY ROGER C. ALPERIN[1] AND PETER B. SHALEN[1]

Our main result provides necessary and sufficient conditions for a finitely-generated subgroup of $GL_n(\mathbf{C})$, $n > 0$, to have finite virtual cohomological dimension. A group has finite virtual cohomological dimension (VCD) if it has a subgroup of finite index which has finite cohomological dimension; this dimension is, in fact, the same for all torsion-free subgroups of finite index. It is, of course, necessary for a group Γ with $\mathrm{VCD}(\Gamma) < \infty$ to have torsion-free subgroups of finite index; this is guaranteed in the case of finitely-generated linear groups by a well-known result of Selberg which extends ideas of Minkowski.

A subgroup of $GL_n(\mathbf{C})$ is called unipotent if it is contained in a conjugate of the group of upper triangular matrices with all diagonal entries equal to one. Any unipotent subgroup is nilpotent; hence, a finitely-generated unipotent subgroup is polycyclic and torsion-free. It is well known that a polycyclic group has finite cohomological dimension if and only if it is torsion-free; moreover, the cohomological dimension is the same as the Hirsch rank. For a solvable group Γ with solvable series, $1 = \Gamma_n < \Gamma_{n-1} < \cdots < \Gamma_1 = \Gamma$, the Hirsch rank, $h(\Gamma) = \Sigma_{i=1}^{n-1} \dim_{\mathbf{Q}}(\Gamma_i/\Gamma_{i+1} \otimes \mathbf{Q})$, is independent of the choice of solvable series; thus, for a polycyclic group Γ, $h(\Gamma)$ is the number of infinite factors in a normal series with cyclic quotients.

We announce our main result.

THEOREM. *Let A be a finitely-generated integral domain of characteristic zero. A group $\Gamma \subset GL_n(A)$, $n > 0$, has finite VCD if and only if there is a finite upper bound on the Hirsch ranks of its finitely-generated unipotent subgroups.*

We obtain easily the following curious corollary.

COROLLARY 1. *Every finitely-generated subgroup of the unitary group $U_n(\mathbf{C})$, $n > 0$, has finite virtual cohomological dimension.*

The following result is immediate; it, however, was original motivation for our Theorem.

Received by the editors December 10, 1980.

1980 *Mathematics Subject Classification*. Primary 20J05; Secondary 20H20.

[1]Research supported by the National Science Foundation.

COROLLARY 2 (SERRE [3, THÉORÈME 5]). *Every finitely-generated subgroup of* $GL_n(\mathbf{Q})$, $n > 0$, *has finite virtual cohomological dimension.*

Ralph Strebel has suggested, as a consequence of our Theorem, that we generalize certain results of Bieri. Before mentioning that generalization, we require the following corollary.

COROLLARY 3. *Let F denote a field of characteristic zero. If Γ is a finitely-generated subgroup of $GL_n(F)$, $n > 0$, with center Z then Γ has finite VCD if and only if Z and Γ/Z have finite VCD.*

A group Γ is said to be of type FP if the trivial Γ-module $\mathbf{Z}$ has a finite resolution by finitely-generated projective $\mathbf{Z}\Gamma$ modules. Combining our Corollary 3 with the methods of Bieri [2] we obtain the following as an immediate consequence.

COROLLARY 4. *If Γ is of type FP and has a faithful linear representation over a field of characteristic zero then the center of Γ is finitely generated.*

The proof of our main theorem involves the action of linear groups on the Tits' buildings for discretely-valued fields. This ingredient already occurs in Serre [3]. Serre's application to groups of type FA [4, Proposition 2] was carried further by Bass [1, Theorem 6.5] in describing finitely-generated subgroups of $GL_2(\mathbf{C})$. Inspired by this we have shown that (with the notation of the Theorem) there are finitely many valuations $v_1, \ldots, v_m$ of the quotient field of A such that $A \cap \mathcal{O}_{v_1} \cap \cdots \cap \mathcal{O}_{v_m}$ is the ring of integers in a number field K. This is used to produce an action of Γ on a contractible cell complex which is a product of finitely many Tits' buildings, such that the stabilizer of each cell consists of matrices whose characteristic roots are algebraic integers in an extension of K having bounded degree. Under the hypothesis of the Theorem, one can bound the virtual cohomological dimensions of these stabilizers by representing them as discrete subgroups of Lie groups.

A result due to Quillen [3, Proposition 2] then implies that Γ has finite VCD. The details of proof will appear elsewhere; the techniques can be further refined to give a theory of hierarchies for matrix groups which is analogous to the Haken-Waldhausen theory for 3-manifolds.

We would like to thank Hyman Bass and Ken Brown for their encouragement throughout the stages of development of these results.

BIBLIOGRAPHY

1. H. Bass, *Groups of integral representation type,* Pacific J. Math. **86** (1980), 15–52.

2. R. Bieri, *A connection between the integral homology and the centre of a rational linear group,* Math. Z. **170** (1980), 263–266.

3. J.-P. Serre, *Cohomologie des groupes discrets*, Prospects in Mathematics, Princeton Univ. Press, Princeton, N. J., 1971.

4. ———, *Amalgames et points fixes*, Proc. Second Internat. Conf. on the Theory of Groups (Australian Nat. Univ., Canberra, 1973), Lecture Notes in Math., Vol. 372, Springer, Berlin, 1974, pp. 633–640.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA 73019

DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TEXAS 77001