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ORDINARY $RO(G)$-GRADED COHOMOLOGY

BY G. LEWIS, J. P. MAY, AND J. McCLURE

Let G be a compact Lie group. What is the appropriate generalization of singular cohomology to the category of G-spaces X? The simplest choice is the ordinary cohomology of $EG \times_G X$, where EG is the total space of a universal principal G-bundle. This Borel cohomology [1] is readily computable and has many applications, but is clearly inadequate for such basic parts of G-homotopy theory as obstruction theory. Another choice is Bredon cohomology [2], as generalized from finite to compact Lie groups by several authors. This gives groups $H_G^n(X; M)$ for $n \geq 0$ and for a "coefficient system" M. Here M is a contravariant functor from the homotopy category of orbit spaces G/H and G-maps to the category Ab of Abelian groups. (Subgroups are understood to be closed.) Bredon cohomology is adequate for obstruction theory. For finite G, Triantafillou has used it to algebraicize rational G-homotopy theory [11], and she and two of us have used it to set up the foundations of the theory of localization of G-spaces for general G [8]. When M is constant at an Abelian group A, written $M = \mathbf{A}$, we have

$$(*) \qquad H_G^n(X; \mathbf{A}) = H^n(X/G; A).$$

In particular, we have

$$H_G^n(EG \times X; \mathbf{A}) = H^n(EG \times_G X; A).$$

Thus Bredon cohomology generalizes Borel cohomology.

Nevertheless, we feel that these theories do not comprise the full equivariant generalization of ordinary singular cohomology. The full theory should build in the interplay relating the Burnside ring $A(G)$, the real representation ring $RO(G)$, and G-homotopy theory. Any cohomology theory must be "stable". Bredon cohomology is only stable in the classical sense that

$$\widetilde{H}_G^n(X; M) = \widetilde{H}_G^{n+q}(\Sigma^q X; M)$$

for a based G-space X (with basepoint a fixed point). A fully equivariant theory E^* should allow groups $E^v(X)$ for all G-representations V; $E^n(X)$ should be the special case of the trivial representation R^n. If SV denotes the 1-point compactification of V and $\Sigma^v X$ denotes $X \wedge SV$, we should have

$$\widetilde{E}^v(X) = \widetilde{E}^{v \oplus w}(\Sigma^w X).$$

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Such $RO(G)$-graded cohomology theories have many advantages. They admit powerful splitting theorems that greatly simplify the task of computation; see e.g. [3]. They come equipped with transfers for G-bundles and G-fibrations with compact fibers [10], [13]. They are essential to the specification of a meaningful notion of orientability of G-bundles and spherical G-fibrations and are therefore essential to a meaningful formulation of even such a basic notion as Poincaré duality [15].

Known examples of $RO(G)$-graded theories include K-theory, various cobordism theories, and theories produced by equivariant infinite loop space theory, a comprehensive treatment of which is given by Hauschild, May, and Waner [6]. The last example includes equivariant algebraic K-theory of rings, which is defined and computed for finite fields by Fiedorowicz, Hauschild, and May [5]. It also includes spherical G-fibration theory, which is used by McClure [9] to prove the sharpest form of the equivariant Adams conjecture and to study the groups $JO_G(X)$.

Ordinary cohomology is missing from this list. For finite G, Waner [14] has used the deepest form of equivariant infinite loop space theory (which also constructs $RO(G)$-graded theories from Galois extensions with Galois group G [6]) to construct ordinary $RO(G)$-graded cohomology theories. However, at this writing all forms of equivariant infinite loop space theory are strictly limited to finite groups, and such a construction is obviously unsatisfactory for something as basic and presumably elementary as ordinary cohomology!

For arbitrary compact Lie groups G and appropriate coefficient systems M, we have extended Bredon cohomology to an $RO(G)$-graded theory $H_G^*(X;M)$. For finite G, M must extend to a Mackey functor as defined by Dress [4]. For general G, we have invented the appropriate notion of a Mackey functor.

As a first application, we have the following simple proof of a deep unpublished theorem of Oliver.

THEOREM. *Let X be a G-space of the homotopy type of a G-CW complex and let H be a (closed) subgroup of G. Let R be any commutative ring. Then there exists a natural transfer homomorphism*

$$\tau\colon H^n(X/H;R) \longrightarrow H^n(X/G;R) \quad \textit{for } n \geq 0$$

such that $\tau \cdot \pi^*$ is multiplication by the Euler characteristic $\chi(G/H)$, where π: $X/H \longrightarrow X/G$ is the map given by inclusion of orbits.

PROOF. By $(*)$ above and by a change of groups isomorphism, we have a commutative diagram

$$\begin{array}{ccc} H_G^n(X;\mathbf{R}) & \cong & H^n(X/G;R) \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi^*} \\ H_G^n(G/H \times X;\mathbf{R}) & \cong & H^n(X/H;R). \end{array}$$

Here π on the left is the projection $G/H \times X \longrightarrow X$, which is obviously a G-bundle. This much precedes our theory, requiring only standard facts about Bredon cohomology. We have checked that $\mathbf{R}$ is the underlying coefficient system of a Mackey functor. Thus the groups on the left are terms in an $RO(G)$-graded cohomology theory. We therefore have a transfer homomorphism connecting them. On a technical note, we have a new treatment of transfer for G-bundles with compact fibres which requires no finiteness condition on the base space. The Euler characteristic formula drops out immediately from general properties of the transfer together with specific properties of the relevant Mackey functor.

REMARKS (i). We are using singular cohomology, wheras Oliver uses Čech cohomology and totally different techniques.

(ii) If H has finite index in G, the result is easy. Here Illman [7] already gave a transfer adequate for our proof.

(iii) The G-CW homotopy type hypothesis on X can be weakened. By Waner's results [12], it is not very restrictive in any case.

Our construction of $H_G^*(X; M)$ is based on use of the equivariant stable category constructed by the first two authors. Let $U = \bigoplus V_i^\infty$, where V_i^∞ is the sum of countably many copies of V_i and $\{V_i\}$ runs through a set of representatives for the irreducible real representations of G. A G-spectrum E is a collection of based G-spaces EV indexed on the finite dimensional invariant subspaces V of U together with G-homeomorphisms $EV \cong \Omega^W E(V + W)$ for V orthogonal to W. Maps $E \longrightarrow F$ are collections of G-maps $EV \longrightarrow FV$ compatible with the given homeomorphisms. Homotopies are families of maps parametrized by the unit interval. A map is a weak equivalence if each fixed point map $(EV)^H \longrightarrow (FV)^H$ is a weak equivalence. The stable category HS_G is obtained from the homotopy category of G-spectra by adjoining formal inverses to the weak equivalences. There is a notion of a G-CW spectrum, and HS_G is equivalent to the homotopy category of G-CW spectra and cellular maps. HS_G has all the good formal properties familiar from the nonequivariant context, and its suspension functor Σ^v is an equivalence. There is a stabilization functor Σ^∞ from based G-spaces to G-spectra, and $\Sigma^\infty X \simeq X \wedge S$ where S is the sphere G-spectrum $\Sigma^\infty S^0$.

A G-spectrum E determines an $RO(G)$-graded theory on G-spectra Y via $E^v(Y) = [Y, \Sigma^v E]_G$. By restriction to $Y = \Sigma^\infty(X^+)$, there results an $RO(G)$-graded theory on G-spaces X, where $X^+ = X \amalg \{*\}$. Here $V \in RO(G)$, desuspension in HS_G allowing the interpretation of $\Sigma^v E$.

Let $\mathcal{O}$ be the full subcategory of HS_G with objects $G/H^+ \wedge S$. We define a Mackey functor to be a contravariant additive functor $M\colon \mathcal{O} \longrightarrow \mathrm{Ab}$. This is (most unobviously!) equivalent to the usual notion when G is finite. By restriction to maps $f^+ \wedge 1$ for G-maps $f\colon G/H \longrightarrow G/K$, a Mackey functor determines an underlying coefficient system. For a G-spectrum Y and integer n, define a Mackey functor $\pi_n Y$ by

$$(\pi_n Y)\,(G/H^+ \wedge S) = [G/H^+ \wedge S^n,\, Y]_G, \quad \text{where } S^n = \Sigma^n S.$$

For example, if $Y = S^n$ the right side is the Burnside ring $A(H)$. For a G-CW spectrum Y with n-skeleton Y^n, define $\mathbf{C}_n Y = \pi_n(Y^n/Y^{n-1})$. Y^n/Y^{n-1} is a wedge of G-spectra of the form $G/H^+ \wedge S^n$. Then $\mathbf{C}_* Y$ is a complex in the Abelian category of Mackey functors. We can form the cellular cochain complex

$$C^*(Y; M) = \mathrm{Hom}_{\mathcal{O}}(\mathbf{C}_* Y, M)$$

and pass to homology to obtain a Z-graded cohomology theory $H^*(Y; M)$ on G-spectra. Its 0th term is represented by an Eilenberg-Mac Lane spectrum $K(M, 0)$. The $RO(G)$-graded theory on G-spectra determined by $K(M, 0)$ extends the Z-graded cellular theory we started with and gives the desired $RO(G)$-graded extension of Bredon cohomology with coefficients in M.

We have a dual construction of $RO(G)$-graded homology theories $H_*(Y; N)$ with coefficients in covariant functors N: $\mathcal{O} \longrightarrow$ Ab. For finite G, $\mathcal{O}$ is self-dual and the two kinds of coefficient systems are equivalent. For general G, $\mathcal{O}$ is not self-dual and a quite different kind of Eilenberg-Mac Lane spectrum $K(N, 0)$ represents these homology theories.

We have obtained change of groups isomorphisms, universal coefficients spectral sequences, Künneth theorems, Green functors and products, Atiyah-Hirzebruch spectral sequences, and so forth for these new theories. Details will appear in due course. Computations are work in progress.

Incidentally, we have also proven that, for finite G, the completed Burnside ring $A(G)$ has defect set the p-Sylow subgroups of G. In particular, this implies that the Segal conjecture, which asserts that $\hat{A}(G)$ is isomorphic to the 0th stable cohomotopy group of BG, is true for all finite groups G if it is true for all p-groups.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637